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Mathematical Models and Minimization Methods of Works Time and Costs on the Project

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Abstract
There is a wide class of planning tasks in which possible technologies of works execution and their combinations can be succeeded by the alternative variants. These variants can be represented as network models. The dependences between scope, time and cost of works execution are discrete. The goal of this work is to create effective methods for these tasks solution.
The mathematical model and minimization method of project execution time on the project with possible funding constraints during different periods and with given alternative variants of the network models fragments are offered in this paper.
For project cost minimization problems solution the mathematical model and minimization method of project works costs with project works execution terms constraints and given alternative variants of the network models fragments are offered.
The offered models are dynamic with Boolean variables with algorithmic criterion function or constraints. The developed methods are related to the group of implicit enumeration methods.

Keywords
Mathematical models, Time, Cost, Project, Network models optimization

1. Background and Object of the Paper
Today attention is being increasingly focused on extension of the interrelation in project management knowledge areas, such as: time, cost, scope management and others.
During the works plans development, as a rule, it is necessary to take into account restrictions on availability different kinds of resources. The given problems concerns to the time scheduling at the limited resources. Two groups of methods were developed for their solution: heuristic (approximate) (D.I. Golenko (1968), D. Phillips, A. Garcia-Diaz (1984), V.I. Voropaev, J.D. Gelrud (2006)) and methods of the optimum plans reception (exact methods).

Exact methods of the time scheduling problem solution under resource restrictions divided into two categories:
– the methods based on linear and nonlinear programming (E.G. Davydov (1990), D. Phillips, A. Garcia-Diaz (1984);
– the methods in which basis search or reduced search of the variants lays (D. Phillips, A. Garcia-Diaz (1984)).
The methods aiming to drawing up of the planned schedule, minimizing expenses by the set project duration are known (D.I. Golenko (1968), D. Phillips, A. Garcia-Diaz (1984), V.I. Voropaev, J.D. Gelrud (2006)).

However, well-known specified problems solution methods do not assume consideration of alternative variants execution of the complexes works or project stages, represented in the network models form and related with many previous and subsequent works.

The object of the work is the creation of the mathematical model and minimization method of the project execution time with possible financing restrictions at separate stages of the project and the creation of the mathematical model and minimization method of the project cost with project works execution time restrictions. Thus it is supposed, that alternative variants of the works complexes execution are set in the network fragments form, hardly connected with the previous and subsequent project works.

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The project scope is set in the network model form, in which operations are interpreted as network nodes, and arches describe technological interrelations between the operations. For each node we will set execution time of operation, and also needed financial resources.

2. Mathematical Model and Minimization Method of the Project Execution Time with its Cost Restrictions

The value of the criterion function is equal to $T_{project} = \varphi(x_{jh})$, it represents duration of a critical path in the network model, describing the project.

Model is of the form:

$$T_{project} = \varphi(x_{jh}) \to \min_{x_{jh}},\ j = 0,1,...,M_h,\ h = \overline{1,H}; \quad (1)$$

$$S_h = S_{h-1} + K_h - w_{j,h};$$

$$S_h \geq 0,\ h = \overline{1,H}; \quad (2)$$

$$x_{jh} \in \{0,1\},\ j = 0,1,...,M_h,\ h = \overline{1,H}; \quad (3)$$

where $T_{project}$ - duration of all project execution operations;
M_h — quantity of the operations execution variants at the h stage, $h = \overline{1,H}$;
h — number of the operations execution stage;
H — the maximum quantity of stages;
x_{jh} — Boolean variable, which is unit value if j-th variant of the operations execution at h-th stage is carried out, and is zero value otherwise;

The value of the criterion function $T_{project} = \varphi(x_{jh})$ calculated as a duration of the critical path in the network model $G = \{A, Z, \tau, W\}$, where
G — network model of the project operations;
A — network node set,
$A = \{a_{ih^j}\}$, $i = \overline{1,n_j}$, $h = \overline{1,H}$, $j = 0,1,...,M_h$,
where a_{ih^j} - i-th operation which is carried out at h-th stage in j-th variant (alternative) of the network model. The variant $j = 0$ is a network model of the base project realization variant;
n_j — operations quantity in j-th variant of the network model.
Z — set of the directed arches,

$$Z = \{z_{ih^j,kp^f}\},\ i,k = \overline{1,n_j},\ h,p = \overline{1,H},\ j = 0,1,...,M_h,\ f = 0,1,...,M_p,$$

where z_{ih^j,kp^f} - the arch which moves out of the node i at h stage of an alternative variant j and penetrates into node k at p stage of an alternative variant f; $i \neq k$ $p \geq h$; τ - set of the operation execution time in nodes,

$$\tau = \{\tau_{ih^j}\},\ i = \overline{1,n_j},\ h = \overline{1,H},\ j = 0,1,...,M_h;$$

W – set of the operations execution costs in the network,
$W = \{w_{ih^j}\}$, $i = \overline{1,n_j}$, $h = \overline{1,H}$, $j = 0,1,...,M_h$,

where w_{ih^j} - execution cost of i-th operation at h-th stage for j-th operation execution variant;
S_0 – money resources allotted to the project execution before its beginning;
S_h – the rest of money resources after works execution at h-th stage;
K_h – volume of money resources allotted at h-th stage;

w_{jh} - operations execution cost of j-th network model variant at h-th stage (can consist of several operations costs).

Restriction (2) assumes that in a course of the project there should not be any financial debts.

The offered model is a dynamic with Boolean variables with algorithmic criterion function with analytical restrictions.
For the problems (1-3) solution the method of the project terms minimization with its cost restrictions is offered, which is concerning to the group of implicit search methods, which is development of the method offered earlier in the work (I.V. Kononenko, O.V. Iemelyanova, A.I. Gricay (2007)).

Let us assume, that alternative variants of the network model can concern to one works execution stage or to several stages. We will describe the information preparation for the worked out method in the form of the consecutive stage.
1. To describe and represent variants of the project execution in the form of network model. To set duration and cost of the operations.
2. To define interrelation of various stages variants, to comprise the arches matrix, containing information of the previous stage last works interrelations with the works of subsequent stage.
3. To carry on the analysis which purpose consists in revealing variants which cover a couple of stages. If some variant covers more than one stage these stages will be united in one.
4. To calculate the bottom borders of the operations execution durations at each h-th stage $h = \overline{1,H}$.

The calculation of the bottom border values assumes execution of some following actions:
4.1 For each of the stages $h = \overline{1,H}$ enter the logic tops of start and end S (start) and T (target.)
4.2. For each of the stages $h = \overline{1,H}$ calculate by the critical path method time of the determined stage execution for all alternatives.
4.3 For each of the stages $h = \overline{1,H}$ choose minimal execution time.
Set of the chosen minimal terms will be $t_{\min} = \left\{t_{\min_h}\right\}_{h=1}^{H}$.
5. To calculate w_{jh} , $j = 0,1,...,M_h$, $h = \overline{1,H}$.
The input information which is needed for the method realization:
G - network model;
H - a maximum stages quantity;
$\{K_h\}_{h=1}^{H}$ - set of the allotted money recourses volumes for each stage;
$\{M_h\}_{h=1}^{H}$ - set of the most possible variants at each stage;
$t_{\min} = \left\{t_{\min_h}\right\}_{h=1}^{H}$ - set of the minimal terms of the project stages execution, calculated prior to the beginning of an iterative part of the method. Variables:
f^* - the record value;
f - current value of criterion function;
h - number of the stage;
S_h - expenses of a stage h ;
j_h - number of the variant at stage h. Value j_h=0 means that the base variant will be chosen;
t_h - time of the operations execution at all stages from 1-st up to h-th inclusive;
T'_{pr_h} - bottom border for the time of all subsequent stages after h execution which represents the sum of a kind:
$T'_{pr_h} = t_{\min_{h+1}} + ... + t_{\min_H}$,
where time $t_{\min_{h+1}}$, ... , $t_{\min_H}$ calculated in the way described earlier.
The solution result :
W_H - the required solution, set of the chosen variants j at all H stages;
We will analyze the offered method of the project execution terms minimization on its cost restrictions:
1. Suppose f^*:=+∞, f:=0, h:=1; $S^* := 0$, $t_{\min} := \left\{t_{\min_h}\right\}_{h=1}^{H}$.
2. Accept j_h:=1.
3. Check performance of the problem restrictions on the stage h:
$S_h = S_{h-1} + K_h - w_{j,h}$;

$S_h \geq 0$,
if restriction is not carried out, we pass to the step 7.

4. We determine t_h by calculation of the network critical path duration from the first stage up to stage *h* inclusive. For this purpose the fictitious top "finish" designating the end of all operations of *h*-th stage is entered.
We believe $f := t_h$. We calculate $T'_{\delta r_h} = t_{\min_{h+1}} + ... + t_{\min_H}$. If $f + T'_{\delta r_h} \geq f^*$, we pass to the step 7.

5. By $h < H$ the following stage of the project is analyzed, i.e. *h*: = *h*+1 and come back to the step 2.

6. To the volume *f** we appropriate new value *f**: = *f* also we fix set $W_H := \{ j_h \}_{h=1}^{H}$. We reduce *f* in the following way $f := t_{H-1}$.

7. By $j_h < M_h$ the following variant is considered, i.e. $j_h := j_h + 1$ and we pass to the step 3.

8. By $h > 1$ we pass to the previous stage, i.e. $h := h - 1$ and we change the value $f := t_{h-1}$. We take from memory j_h , and pass to the step 7. By *h*=1 and $W_H = \{\varnothing\}$ the problem has no solution, otherwise the optimum solution is received. Thus $T_{project} = f^*$.

3. Mathematical Model and Minimization Method of the Project Execution Cost with its Terms Restrictions

The value of the criterion function represents the project working out and realization expenses. Restrictions are time of the project execution, financing volumes of the separate project works stages and the project as a whole.

Model is of the form:

$$\sum_{h=1}^{H} \sum_{j=1}^{M} w_{jh} x_{jh} = F \rightarrow \min_{x_{jh}} \quad ;(4)$$

$$w_{jh} = \sum_{i=1}^{n_j} w_{ih^j} \quad ; \qquad (5)$$

$$T \leq T^{set}, \quad T = \varphi(x_{jh}); \qquad (6)$$

$$S_h = S_{h-1} + K_h - w_{jh}\,,\ S_h \geq 0, \quad h = \overline{1, H}; \qquad (7)$$

$$x_{jh} \in \{0,1\}, \quad j = \overline{1, M}, \quad h = \overline{1, H}; \qquad (8)$$

$$\sum_{j=1}^{M} x_{jh} \leq 1, \ \ h = \overline{1, H}. \qquad (9)$$

If in the network model there are sites which have no alternatives or these alternatives are not examined, so the given sites are named by the first model variant at the corresponding stage.

For checking condition (6) duration of the critical path in the network model $G = \{ A, Z, \tau, W \}$ is calculated.

Restriction (7) assumes that while project realization there should not be any financial debts.

Expression (8) characterizes restriction, that at each stage h it is possible to begin no more than one variant realization.

The offered model is a dynamic with Boolean variables with analytical criterion function with algorithmic and analytical restrictions.

For the problem solution the method of the project cost minimization with its time restrictions is offered, which is concerning to the group of implicit search methods. The difference of the offered method consists in the way of time restrictions execution estimating and the bottom border for the project cost estimating.

The information preparation for the given method is similar to the method of the project execution time minimization.

4. Example of Calculation

As an example the network model which reflects the scope of the project is examined. The given project consists of three stages. At each stage alternative variants of the works complexes execution in the network fragments form are set. For each examined network node, the operation execution durations are set, and also volume expenses, which are needed for each its execution.

After the settled accounts with the minimization method of the project execution times following results have been received:

- Time of the project realization takes 118 days
- Cost of the project execution is 5,55 million USD.

After the settled accounts with the minimization method of the project execution cost following results have been received:

- Cost of the project realization takes 5,2 million USD
- Time of the project execution is 140 days.

5. Conclusion

The mathematical model and minimization method of the project execution time with the possible financing restrictions at the separate stages of the project and also mathematical model and minimization method of the project execution cost with the project works execution time restrictions are offered. The given models and methods allow to solve planning problems in conditions when alternative variants are set in the network fragments form with the complex interrelations with subsequent and previous works.

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