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ON A PROPERTY OF PAIRS OF ALMOST PERIODIC ZERO SETS

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Whenever all differences of zeros of two holomorphic almost periodic functions in a strip form a discrete set, then both functions are infinite products of periodic functions with commensurable periods. In particular, the result is valid for some classes of Dirichlet series.

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Доказано, что если множество разностей нулей двух голоморфных почти периодических функций в полосе дискретно, то обе функции являются бесконечным произведением периодических функций с соизмеримыми периодами. В частности, результат справедлив для некоторых классов рядов Дирихле.

It was proved in [1] that each quasipolynomial

$$Q(z)=\sum_{n=1}^{N}a_ne^{\lambda_n z},\quad \lambda_n\in\mathbb{R},\quad a_n\in\mathbb{C},\tag{1}$$

with a discrete set of differences of its zeros is periodic up to a multiplier without zeros, consequently it has the form

$$Q(z)=Ce^{\beta z}\prod_{k=1}^{N}\cosh(\omega z+b_k),\quad \beta,\omega\in\mathbb{R},\quad \mathbb{C},\ b_k\in\mathbb{C}.\tag{2}$$

The result is also valid for infinite sums

$$S(z)=\sum_{n=1}^{\infty}a_ne^{\lambda_n z},\quad \lambda_n\in\mathbb{R},\quad a_n\in\mathbb{C},\tag{3}$$

under conditions

$$\sum_{n=1}^{\infty}|a_n|<\infty,\quad \lambda_1=\sup_n\lambda_n<\infty,\quad \lambda_2=\inf_n\lambda_n>-\infty,\quad a_1a_2\neq 0.\tag{4}$$

Note that zeros of functions (1), (3) are located in a vertical strip of a finite width.

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In 1949 M. G. Krein and B. Ja. Levin (see [5], also [6], Ch. 6, p. 2 and Appendix 6) introduced and studied the class Δ of entire almost periodic functions of exponential growth with zeros in a horizontal strip of a finite width. In particular, if the sum $S(z)$ in (3) satisfies condition (4), then $S(iz)$ belongs to Δ. In fact, representation (2) was obtained in [1] for functions from Δ with a discrete sets of differences of zeros (of course, one should replace $e^{\beta z}$ by $e^{i\beta z}$ and $\cosh(\omega z + b_k)$ by $\cos(\omega z + b_k)$).

The similar phenomenon takes place for a pair of functions when the set of differences of their zeros is discrete ([3]). For example, let Q_1, Q_2 be entire functions of form (3) under conditions (4). If the set $\{z - w,\ Q_1(z) = 0,\ Q_2(w) = 0\}$ is discrete, then the both functions have form (2) with the same ω and possibly different C, β, N, b_k. In particular, in the case $Q_2(-z) = Q_1(z) = Q(z)$ we get representation (2) for any function (3) with the discrete set $\{z + z',\ Q(z) = Q(z') = 0\}$.

The proof of the above results are based on the property of zeros $\{z_j\}$ of functions $f \in \Delta$ ([6], Appendix 6, p. 2) to be almost periodic in the sense of the following definition.

Definition 1. A zero set $\{z_j\}$ is *almost periodic* if for any $\varepsilon > 0$ there is a relatively dense set $E_\varepsilon \subset \mathbb{R}$ such that for each $\tau \in E_\varepsilon$ there exists a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ with the property

$$\sup_j |z_j + i\tau - z_{\sigma(j)}| < \varepsilon. \tag{5}$$

Recall that a set $E \subset \mathbb{R}$ is relatively dense if there exists $L < \infty$ such that $E \cap [a, a+L] \neq \varnothing$ for any $a \in \mathbb{R}$.

In a sense, the above definition is applied to zero sets in a *closed* vertical strip. In the present paper we investigate holomorphic almost periodic functions and almost periodic sets in an open strip, in particular, in the complex plane and in half-planes.

Definition 2. A continuous function $f(z)$ in the strip $S = \{z\colon a < \operatorname{Re} z < b\}$, $-\infty \le a < b \le \infty$ is *almost periodic*, if for any substrip S_0, $\overline{S_0} \subset S$, and any $\varepsilon > 0$ there is a relatively dense set $E_{\varepsilon,S_0} \subset \mathbb{R}$ such that for any $\tau \in E_{\varepsilon,S_0}$

$$\sup_{z \in S_0} |f(z + i\tau) - f(z)| < \varepsilon. \tag{6}$$

A typical example of an entire almost periodic function is sum (3) under conditions

$$\sum_{n=1}^{\infty} |a_n| < \infty, \quad \sup_n \lambda_n < \infty, \quad \inf_n \lambda_n > -\infty.$$

Note that a holomorphic function $f(z)$ in the strip $S = \{z\colon a < \operatorname{Re} z < b\}$, $-\infty \le a < b \le \infty$ is almost periodic if and only if there exists a sequence of quasipolynomials Q_n of form (1) such that for any substrip S_0, $\overline{S_0} \subset S$,

$$\sup_{z \in S_0} |f(z) - Q_n(z)| \to 0, \quad n \to \infty,$$

(see [4], item 8).

In order to take into account multiplicities of zeros, we use the term *divisor* instead of zero set. Namely, a divisor Z in a domain D is the mapping $Z\colon D \to \mathbb{N} \cup \{0\}$ such that $|Z| = \operatorname{supp} Z$ is a set without limit points in D. In other words, a divisor in D is the sequence of points $\{z_j\} \subset D$ without limit points in D such that every point of D appears at most finite times in the sequence. If $Z(z) \leq 1$ for all $z \in D$, we identify Z and $|Z|$. The divisor of zeros of a holomorphic function f in D is the map Z_f such that $Z_f(a)$ equals the multiplicity of zero of the function f at the point a.

The following definition appeared at first in [8].

Definition 3 ([2], [8])**.** A divisor $Z = \{z_j\}$ in a strip $S = \{z\colon a < \operatorname{Re} z < b\}$, $-\infty \leq a < b \leq \infty$, is called *almost periodic* if for any $\varepsilon > 0$ and any substrip S_0, $\overline{S_0} \subset S$, there is a relatively dense set

$$E_{\varepsilon,S_0} = \{\tau \in \mathbb{R}\colon z_j \in S_0 \vee z_{\sigma(j)} \in S_0 \Longrightarrow |z_j + i\tau - z_{\sigma(j)}| < \varepsilon\}, \tag{7}$$

where $\sigma = \sigma_\tau$ is a suitable bijection $\mathbb{N} \to \mathbb{N}$.

Remark. If we take $\sigma^{-1}(j)$ instead of j in (7), we get

$$z_j \in S_0 \vee z_{\sigma^{-1}(j)} \in S_0 \Longrightarrow |z_j - i\tau - z_{\sigma^{-1}(j)}| < \varepsilon. \tag{8}$$

Theorem 1 ([2])**.** *a) If f is a holomorphic function in a strip S with the almost periodic modulus, then the divisor of zeros of f is almost periodic.*

b) For any almost periodic divisor Z in a strip S there is the holomorphic function f in S with the almost periodic modulus such that $Z = Z_f$

c) For any almost periodic divisor Z in a strip S such that $|Z| \cap S' = \varnothing$ for some open substrip $S' \subset S$ there is the holomorphic almost periodic function f in S such that $Z = Z_f$.

There exists an almost periodic divisor in the plane with a discrete set of differences of their points that is not periodic.

Example 1. Let

$$Z = \{z_{n,k} = i2^{nk} + 2^k,\ n \in \mathbb{Z},\ k \in \mathbb{N}\}$$

be a discrete set in $\mathbb{C}$. It is easy to see that the set is almost periodic, differences of its points form a discrete set, but Z is a countable union of periodic sets with different commensurable periods.

In our article we prove the following theorem:

Theorem 2. *Let Z, W be almost periodic divisors in the strip $S = \{z\colon a < \operatorname{Re} z < b\}$, $-\infty \leq a < b \leq \infty$. If*

- *a) $|Z| \cap S' = \varnothing$ for some substrip $S' \subset S$,*
- *b) for any substrip S_0 of a finite width, $\overline{S_0} \subset S$, the set $\{z - w,\ z \in |Z| \cap S_0,\ w \in |W| \cap S_0\}$ is discrete,*

then Z, W are at most countable sums of periodic divisors with commensurable periods.

Show that condition a) is essential.

Example 2. Let $S = \{z\colon |\operatorname{Re} z| < 1\}$, $Z = \{z_{m,n} = (m + in)e^{i\alpha} \in S,\ m, n \in \mathbb{Z}\}$, where $\alpha \in \mathbb{R}$ such that cot α is an irrational number. Clearly, the set of differences of elements of Z is discrete. Let us prove that Z is an almost periodic divisor without any periods.

By Kronecker Lemma (see, for example, [7], Ch.2, §2), for any $\varepsilon > 0$ the inequalities

$$|\exp(2\pi it\cot\alpha) - 1| < \varepsilon, \qquad |\exp(2\pi it) - 1| < \varepsilon,$$

has a relatively dense set of common solutions. Therefore, the inequality

$$|\exp(2\pi im\cot\alpha)-1|<\varepsilon(1+|\cot\alpha|)$$

has a relatively dense set of integer solutions. The latter means that for any δ there exist pairs of integers $(m,n)\in\mathbb{Z}^2$ such that

$$|m\cos\alpha-n\sin\alpha|<\delta, \tag{9}$$

and the set of m with this property is relatively dense. Put

$$E=\{m\sin\alpha+n\cos\alpha\colon |m\cos\alpha-n\sin\alpha|<\delta\}.$$

Let $\tau=\tilde{m}\sin\alpha+\tilde{n}\cos\alpha\in E$. For any $z_{m,n}\in Z$ and $m'=m+\tilde{m},\ n'=n+\tilde{n}$ we have

$$z_{m,n}+i\tau-z_{m',n'}=(m+in)e^{i\alpha}+i\operatorname{Im}[(\tilde{m}+i\tilde{n})e^{i\alpha}]-(m'+in')e^{i\alpha}=-\operatorname{Re}[(\tilde{m}+i\tilde{n})e^{i\alpha}].$$

By (9), $|z_{m,n}+i\tau-z_{m',n'}|<\delta$. Since

$$|\tau-(\cos^2\alpha/\sin\alpha+\sin\alpha)\tilde{m}|=|\tilde{n}\cos\alpha-(\cos^2\alpha/\sin\alpha)\tilde{m}|<\delta|\cot\alpha|,$$

we see that the set E is relatively dense. So, Z is an almost periodic divisor.

Since Z sites in the vertical strip of width 2, we see that any period of Z must have the form $iT,\ T\in\mathbb{R}$. Hence, for some $(n,m),\,(n',m')\in\mathbb{Z}^2,\ (n,m)\neq(n',m')$,

$$(n+im)e^{i\alpha}-(n'+im')e^{i\alpha}=iT,\quad (n-n')+i(m-m')=iT(\cos\,\alpha-i\sin\,\alpha).$$

This equality contradicts our choice of α. Hence any part of Z has no periods.

Our proof of Theorem 2 uses the following lemmas.

Lemma 1. *Let $Z=\{z_n\}$ be an almost periodic divisor in a strip S, S_0 be a substrip, $\overline{S_0}\subset S,\ \varepsilon>0,\ \tau_1,\,\tau_2\in E_{\varepsilon,S_0}$, where E_{ε,S_0} is from Definition 3. Then there is a bijection $\sigma\colon\mathbb{N}\to\mathbb{N}$ such that if $z_j\in S_0$, dist$(z_j,\partial S_0)>\varepsilon$, then*

$$|z_j+i(\tau_1-\tau_2)-z_{\sigma(j)}|<2\varepsilon,\quad |z_j-i(\tau_1-\tau_2)-z_{\sigma^{-1}(j)}|<2\varepsilon. \tag{10}$$

It can be proved easily that the similar assertion is valid for $\tau_1+\tau_2$ as well.

Proof. If z_j satisfies conditions of the lemma, then for some bijection $\sigma_1\colon\mathbb{N}\to\mathbb{N}$ we have $|z_j+i\tau_1-z_{\sigma_1(j)}|<\varepsilon$, therefore, $|\operatorname{Re}(z_{\sigma_1(j)}-z_j)|<\varepsilon$ and $z_{\sigma_1(j)}\in S_0$. By (8), we also get the inequality $|z_{\sigma_2^{-1}\circ\sigma_1(j)}-i\tau_2-z_{\sigma_1(j)}|<\varepsilon$ for some bijection $\sigma_2\colon\mathbb{N}\to\mathbb{N}$. Hence we obtain the first part in (10) with $\sigma=\sigma_2^{-1}\circ\sigma_1$. If we change places of τ_1 and τ_2, we obtain the second inequality in (10). □

Lemma 2. *Let $Z=\{z_n\}$, $W=\{w_m\}$ be almost periodic divisors in a strip S. Then for any $\varepsilon>0$ and any substrip $S_0\subset S$ such that dist$(S_0,\partial S)>\varepsilon$, there is a relatively dense set $E\subset\mathbb{R}$ with the property: for any $\tau\in E$ there are bijections σ_Z, σ_W from $\mathbb{N}$ to $\mathbb{N}$ such that for all $z_j\in|Z|\cap S_0$, $w_r\in|W|\cap S_0$*

$$|z_j+i\tau-z_{\sigma_Z(j)}|<\varepsilon,\quad |w_r+i\tau-w_{\sigma_W(r)}|<\varepsilon, \tag{11}$$

$$|z_j-i\tau-z_{\sigma_Z^{-1}(j)}|<\varepsilon,\quad |w_r-i\tau-w_{\sigma_W^{-1}(r)}|<\varepsilon. \tag{12}$$

Proof. Put $\tilde S = \{z \in S\colon \mathrm{dist}(z, S_0) < \varepsilon/2\}$. By (7) and (8), there is a number L such that any interval of length L contains τ_Z and τ_W with the properties

$$|z_j + i\tau_Z - z_{\sigma_Z(j)}| < \varepsilon/4, \quad |z_j - i\tau_Z - z_{\sigma_Z^{-1}(j)}| < \varepsilon/4 \tag{13}$$

for all $z_j \in |Z| \cap \tilde S$, and

$$|w_r + i\tau_W - w_{\sigma_W(r)}| < \varepsilon/4, \quad |w_r - i\tau_W - w_{\sigma_W^{-1}(r)}| < \varepsilon/4 \tag{14}$$

for all $w_r \in |W| \cap \tilde S$. Here σ_Z, σ_W are some bijections $\mathbb{N} \to \mathbb{N}$.

We may suppose that $N = 2L/\varepsilon$ is an integer. Hence for any $k \in \mathbb{Z}$ there are integers $n(k)$, $m(k)$, $0 \le n(k), m(k) \le N$, such that

$$|kL + n(k)\varepsilon/2 - \tau_Z| < \varepsilon/4, \quad |kL + m(k)\varepsilon/2 - \tau_W| < \varepsilon/4.$$

The differences $n(k) - m(k)$ take at most $2N+1$ values. Choose $k_1, \dots, k_r$, $r \le 2N+1$, such that for any $k \in \mathbb{Z}$ there is k_s with the property $n(k) - m(k) = n(k_s) - m(k_s)$. It follows easily that any interval of length $L(\max_s(|k_s| + 2)$ contains a point of the form

$$\tau = Lk + n(k)\varepsilon/2 - Lk_s - n(k_s)\varepsilon/2 = Lk + m(k)\varepsilon/2 - Lk_s - m(k_s)\varepsilon/2.$$

If we replace τ_Z by $Lk + n(k)\varepsilon/2$ or $Lk_s + n(k_s)\varepsilon/2$ in (13), we obtain that this inequality satisfies with $\varepsilon/2$ instead of $\varepsilon/4$. The same is true if we replace τ_W by $Lk + m(k)\varepsilon/2$ or $Lk_s + m(k_s)\varepsilon/2$ in (14). Applying Lemma 1 with $S_0 = \tilde S$ and $\varepsilon/2$ instead of ε, we see that τ satisfies (11) and (12). □

Proof of Theorem 2. Set a sequence of vertical open substrips S_k, $\overline{S_k} \subset S_{k+1}$, such that

$$|Z| \cap S_1 \ne \varnothing, \quad |W| \cap S_1 \ne \varnothing, \quad S_1 \supset \overline{S'}, \quad \bigcup_k S_k = S.$$

Take any points $z \in |Z| \cap S_1$, $w \in |W| \cap S_1$.

It follows from (11) with $\varepsilon < \min\{\mathrm{dist}(z, \partial S_1), \mathrm{dist}(w, \partial S_1)\}$ that there is $R < \infty$ such that any horizontal strip of the width R contains at least one point $|Z| \cap S_1$ and at least one point $|W| \cap S_1$. Let δ be less than the width of the strip S'. By condition b) of the theorem, for each k the set $\{z - w,\ z \in |Z| \cap S_k,\ w \in |W| \cap S_k\}$ is discrete. Hence there exists $\gamma_k < \min\{1/2, \delta, \mathrm{dist}(S_k, \partial S_{k+1})\}$ such that whenever

$$\begin{gathered} z_n, z_{n'} \in |Z| \cap S_{k+1},\ w_m, w_{m'} \in |Z| \cap S_{k+1}, \quad z_n - w_m \ne z_{n'} - w_{m'}, \\ |\operatorname{Im}(z_n - w_m)| < 2R + 3, \quad |\operatorname{Im}(z_{n'} - w_{m'})| < 2R + 3, \end{gathered}$$

we get $\gamma_k < |(z_{n'} - w_{m'}) - (z_n - w_m)|$. In particular, if we put $w_m = w_{m'}$, then we get $\gamma_k < |z_n - z_{n'}|$ for any $z_n, z_{n'} \in |Z| \cap S_{k+1}$, $z_n \ne z_{n'}$.

Fix $z_n \in |Z| \cap S_k$, and let a number $\tau > 1$ satisfies (11) for Z and W with $\varepsilon = \gamma_k/2$. Then there is a unique $z_{n'} \in |Z| \cap S_{k+1}$ such that $|z_n + i\tau - z_{n'}| < \gamma_k/2$. Indeed, otherwise we obtain

$$|z_{n'} - z_{n''}| \le |z_n + i\tau - z_{n'}| + |z_n + i\tau - z_{n''}| < \gamma_k.$$

Set $T_k = (z_{n'} - z_n)/i$. First suppose that T_k is real, hence, $z_{n'} \in |Z| \cap S_k$.

Let $w_m \in |W| \cap S_k$ be such that $|\operatorname{Im}(w_m - z_n)| < 2R + 2$. By (11), there is a point $w_{m'} \in |W| \cap S_{k+1}$ such that $|w_m + i\tau - w_{m'}| < \gamma_k/2$. Therefore,

$$|(z_n - w_m) - (z_{n'} - w_{m'})| \le |w_m + i\tau - w_{m'}| + |z_{n'} - z_n - i\tau| < \gamma_k.$$

Since

$$|\operatorname{Im}(z_{n'} - w_{m'})| \le |\operatorname{Im}(z_n - w_m)| + |z_n - z_{n'} + i\tau| + |w_m - w_{m'} + i\tau| < 2R + 3,$$

we get $z_n - w_m = z_{n'} - w_{m'}$ due to the choice of γ_k. Therefore, $w_{m'} = w_m + iT_k,\ w_{m'} \in S_k$.

The latter equality takes place for all points $|W| \cap \{w \in S_k\colon \operatorname{Im} z_n - 2R < \operatorname{Im} w < \operatorname{Im} z_n + 2R\}$, in particular, for some w_l such that $\operatorname{Im} z_n + R < \operatorname{Im} w_l < \operatorname{Im} z_n + 2R$. Namely, there is $w_{l'} \in |W| \cap S_k$ such that $w_{l'} = w_l + iT_k$. Let $\zeta \in |Z| \cap S_k$ be any point from the set

$$\{z \in S_k\colon \operatorname{Im} z_n \le \operatorname{Im} z < \operatorname{Im} z_n + 3R\} \subset \{z \in S_k\colon \operatorname{Im} w_l - 2R < \operatorname{Im} z < \operatorname{Im} w_l + 2R\}. \quad (15)$$

By (11), there is a point $\zeta' \in |Z| \cap S_{k+1}$ such that $|\zeta + i\tau - \zeta'| < \gamma_k/2$. Therefore,

$$|(\zeta - w_l) - (\zeta' - w_{l'})| \le |\zeta + i\tau - \zeta'| + |iT_k - i\tau| < \gamma_k.$$

Since $|\operatorname{Im}\zeta - \operatorname{Im} w_l| < 2R$ and

$$|\operatorname{Im}(\zeta' - w_{l'})| \le |\operatorname{Im}(\zeta - w_l)| + |\zeta + i\tau - \zeta'| + |iT_k - i\tau| < 2R + 3,$$

we get $\zeta - w_l = \zeta' - w_{l'}$ due to the choice of γ_k. Therefore, $\zeta' = \zeta + iT_k$.

In particular, there is a point $z_s \in |Z| \cap \{z \in S_k\colon \operatorname{Im} z_n + 2R \le \operatorname{Im} z < \operatorname{Im} z_n + 3R\}$ such that $z_s + iT_k \in |Z|$. Continuing the line of reasoning, we obtain that for all $z \in |Z|$ such that $\operatorname{Im} z \ge \operatorname{Im} z_n$ we get $z + iT_k \in |Z|$ and for all $w \in |W|$ such that $\operatorname{Im} w \ge \operatorname{Im} z_n$ we get $w + iT_k \in |W|$.

If we take $w_l' \in |W| \cap S_k$ such that $\operatorname{Im} z_n - 2R < \operatorname{Im} w_l' < \operatorname{Im} z_n - R$, we also can find $w_{l'}' \in |W|$ such that $w_{l'}' = w_l' + iT_k$. Next, we prove that for any point $\tilde\zeta \in |Z| \cap \{z \in S_k\colon \operatorname{Im} z_n - 3R \le \operatorname{Im} z < \operatorname{Im} z_n\}$ there is $\tilde\zeta' \in |Z| \cap S_k$ such that $\tilde\zeta' = \tilde\zeta + iT_k$. Arguing as above, we show that for all $z \in |Z| \cap S_k$ such that $\operatorname{Im} z \le \operatorname{Im} z_n$ we get $z + iT_k \in |Z|$ and for all $w \in |W| \cap S_k$ such that $\operatorname{Im} w \le \operatorname{Im} z_n$ we get $w + iT_k \in |W|$.

Next, by (12), take for any $z \in |Z| \cap S_k$ a point $z'' \in |Z| \cap S_{k+1}$ such that $|z'' + i\tau - z| < \gamma_k/2$. Then $z'' + iT_k \in |Z| \cap S_k$ and $|(z'' + iT_k) - z| \le |z'' + i\tau - z| + |i\tau - iT_k| < \gamma_k$. Therefore, $z'' + iT_k = z$ and $z - iT_k \in |Z|$ for all $z \in |Z| \cap S_k$. By the same arguments, $w - iT_k \in |W|$ for all $w \in |W| \cap S_k$.

Now suppose that $\operatorname{Im} T_k \ne 0$. Clearly, either $\operatorname{dist}\{z_n, S'\} > \operatorname{dist}\{z_n + iT_k, S'\}$, or $\operatorname{dist}\{z_n, S'\} > \operatorname{dist}\{z_n - iT_k, S'\}$. In the first case the points $z_{n'}$, w_l, $w_{l'}\,\zeta$, $\zeta' \dots$ belong to S_k. Hence, we get $z_n + iMT_k \in S'$ for some integer $M > 0$, that is impossible. In the second case the same arguments show that $z_n + iMT_k \in S'$ for some integer $M < 0$. We get a contradiction in the both cases. Therefore, T_k is real and for any $M \in \mathbb{Z}$ we get $(|Z| \cap S_k) + iMT_k = |Z| \cap S_k$, $(|W| \cap S_k) + iMT_k = |W| \cap S_k$. Thus the restrictions $Z\,|_{S_k}$, $W\,|_{S_k}$ are periodic divisors with period iT_k.

The same arguments work for every $k \in \{1, 2, \dots\}$.

Let iT_k^0 be the minimal common period of $Z\,|_{S_k}$ and $W\,|_{S_k}$. Clearly, $T_k/T_k^0 \in \mathbb{N}$. Besides, since $|Z| \cap S_k \subset |Z| \cap S_m$ for $m > k$, we have $T_m/T_k^0 \in \mathbb{N}$ as well.

Finally, let $Z_1 = Z\,|_{S_1},\ Z_k = Z\,|_{S_k \setminus S_{k-1}},\ W_1 = W\,|_{S_1},\ W_k = W\,|_{S_k \setminus S_{k-1}}$. Then we obtain

$$Z = Z_1 + Z_2 + Z_3 + \dots, \qquad W = W_1 + W_2 + W_3 + \dots. \qquad \square$$

Theorem 2 implies the corresponding result for almost periodic holomorphic functions.

Theorem 3. *Let f, g be almost periodic functions in a strip $S = \{z\colon a < \operatorname{Re} z < b\}, -\infty \le a < b \le \infty$. If*

a) either f, or g has no zeros in an open substrip $S' \subset S$,

b) for any substrip S_0, $\overline{S_0} \subset S$, the set $\{z - w,\ z, w \in S_0,\ f(z) = g(w) = 0\}$ is discrete,

then

$$f(z) = f_0(z) \prod_{k=1}^{\infty} f_k(z), \quad g(z) = g_0(z) \prod_{k=1}^{\infty} g_k(z), \tag{16}$$

where f_0, g_0 are holomorphic almost periodic functions in S without zeros, f_k, g_k, $k \in \{1, 2, \dots\}$, are periodic holomorphic in S with commensurable periods iT_k.

Proof. By Theorem 1a), the divisors Z, W of zeros of f, g, respectively, are almost periodic and satisfy other conditions of the previous theorem. Let S_k, Z_k, W_k, be the same as in the proof of Theorem 2. Suppose that

$$S_k = \{z\colon \eta_k < \text{Re } z < \eta'_k\}, \quad \eta_k < \eta_{k-1}, \quad \eta'_k > \eta'_{k-1}, \ \forall k.$$

For any k there is only a finite number of points $a_1^k, \dots, a_{m_k}^k \in |Z_k| \cap \{z\colon 0 \le \operatorname{Im} z < T_k\}$. Therefore, $|Z_k| = \{a_1^k, \dots, a_{m_k}^k\} + iT_k\mathbb{Z}$. Take $\varepsilon_k^j > 0$, $1 \le j \le m_k$, $3 \le k < \infty$, such that

$$\sum_{j,k} \varepsilon_k^j < \infty. \tag{17}$$

Set

$$h_j(w) = 1 - w \exp(-2\pi a_j^k / T_k), \quad k \in \{1, 2, ...\}, \quad j \in \{1, \dots, m_k\}.$$

Clearly, $h_j(\exp(2\pi z/T_k))$ has the divisor $a_j^k + iT_k\mathbb{Z}$. Now let $k > 2$. Since $\alpha_j^k \in S_k \setminus S_{k-1}$, we see that either $\eta'_{k-1} \le \text{Re } a_j^k$, or $\eta_{k-1} \ge \text{Re } a_j^k$. In the first case, $\log h_j(w)$ is holomorphic on the disc $|w| < \exp(2\pi \operatorname{Re} a_j^k / T_k)$, and there is a polynomial P_j^k such that

$$|\log h_j(w) - P_j^k(w)| < \varepsilon_k^j \quad \text{for } |w| \le \exp(2\pi\eta'_{k-2}/T_k).$$

In the second one, $\log h_j(w)$ is holomorphic on the set $|w| > \exp(2\pi \operatorname{Re} a_j^k / T_k)$, and there is a polynomial P_j^k such that

$$|\log h_j(w) - P_j^k(w^{-1})| < \varepsilon_k^j \quad \text{for } |w| \ge \exp(2\pi\eta_{k-2}/T_k).$$

Put $Q_j(z) = P_j^k(\exp(2\pi z/T_k))$ in the first case, and $Q_j(z) = P_j^k(\exp(-2\pi z/T_k))$ in the second one. We obtain

$$|h_j(e^{2\pi z/T_k}) e^{-Q_j(z)}| < e^{\varepsilon_j^k} \quad \text{for } z \in S_{k-2}. \tag{18}$$

Put

$$f_k(z) = \prod_{j=1}^{m_k} h_j(e^{2\pi z/T_k}),\ k = 1, 2, \quad f_k(z) = \prod_{j=1}^{m_k} h_j(e^{2\pi z/T_k}) e^{-Q_j(z)},\ k > 2.$$

Clearly, $Z_{f_k} = Z_k$. By (17) and (18), the first product in (16) converges in every substrip S_k. The function $f(z)/\prod_{k=1}^{\infty} f_k(z)$ is holomorphic in S, and, by [8], almost periodic in S. In the same way, we obtain the representation of g. □

Remark. It follows easily that for entire almost periodic functions f, g with zeros in a strip $\tilde{S}$ of a finite width one can take $S_1 = \tilde{S}$. Therefore, f, g are periodic functions with the same period up to almost periodic multiplies without zeros. Hence some results of [1], [3] follow from Theorem 3.

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