Stability of stationary regimes in nonlinear systems: analytical and numerical approaches

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Abstract. A stability of stationary regimes in the form of nonlinear normal modes (NNMs) with rectilinear or near-rectilinear trajectories is analysed by using the Ince algebraization when a variable associated with the vibration mode is chosen as the new independent argument. In this case the variational equations are transformed to equations with singular points. Other approach is realized for NNMs with regular or chaotic behavior in time. Namely, a test which is a consequence of the well-known Lyapunov criterion of stability is used. Both approaches are also used in analysis of stability of other stationary regimes, namely, standing or traveling nonlinear waves.

Introduction

The stability of the NNMs [1,2] can be analyzed by different approaches. The Ince algebraization [3] consists of choice of new independent variable associated with the unperturbed solution instead of time. Then the variational equations with periodic coefficients transform to equations with singular points. An advantage of the Ince algebraization is that we do not need in use of the unperturbed solution time-presentation [4]. The NNMs concept can be used not only for periodic vibrations. In particular, the NNMs having smooth trajectories in configuration space and chaotic in time behavior can be found in some non-conservative systems. In particular, such vibration modes are observed in post-buckling forced dynamics of elastic systems that have lost stability under external compressive force. The problem of stability of chaotic modes in a space with a greater dimension has no analytical solutions. Here some test which is a consequence from the classical Lyapunov definition of stability is proposed and used [5]. Both proposed approaches can be used also for other stationary regimes such as traveling or standing nonlinear waves.

The Ince algebraization and stability of nonlinear normal vibration modes

Let the set of variational equations for the NNMs is decomposed to independent equations [1,2]. One of the equations will govern the variations along the rectilinear trajectory, and other ones will refer to the orthogonal directions. The orbital stability of NNM is associated namely with the variations in orthogonal directions. The algebraization by Ince is performed for variational equations by choosing a new independent variable associated with the unperturbed NNM. Then equations with periodic coefficients transform to equations with singular points, following [3,4], together with the relevant change in the stability problem. Singular points here correspond to return points of the NNM. A problem of determination of solutions corresponding to boundaries of the stability/ instability regions is reduced here to Sturm-Liouville problems for functions that are either regular, or have singularity at the mentioned points. Using the Ince algebraization we do not need in use of the time-presentation of the solution which stability is analyzed. In particular, if the system has the potential energy as the even homogeneous function, corresponding variational equations can be presented after the algebraization in the form of the hypergeometric equations. Corresponding “boundary” solutions are the Gegenbauer polynomials. For the system having linear terms and cubic nonlinearity the variational equations transform by the algebraization to Lame equations, where both infinite, and finite number of the instability zones can be determined [1,2]. The Ince algebraization is used for the NNMs stability analysis in the problem of nonlinear absorption and in dynamics of shallow arches [2]. In points of the change of the NNMs stability the bifurcation appears. Examples of forking solutions are presented in Fig. 1 in configuration place, where the return points are equal to $\pm 2a$.

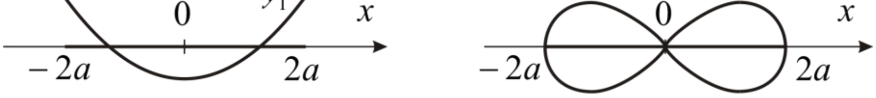

Figure 1: Examples of bifurcation solutions appeared in points of the NNMs stability change.

The proposed approach can be also used in stability analysis of the traveling waves of the Klein-Gordon equation with cubic nonlinearity. Introducing the function which describes the stationary wave as the new independent variable, we can obtain the variational equation with singular points. Then boundaries of the stability/ instability regions in the system parameter space can be obtained.

Stability of nonlinear normal modes with regular or chaotic behaviour in time

Consider the following system that can be obtained by discretization of some nonlinear elastic systems:

$$\begin{aligned} &\ddot{y}_1 + \delta\, \dot{y}_1 - \alpha\, y_1 + \beta\, y^3{}_1 + c\, y_1\, y_2{}^2 = f \cos\omega t, \\ &\ddot{y}_2 + \delta\, \dot{y}_2 + a y_2 + b\, y_2{}^3 + c\, y_2\, y_1{}^2 = 0, \end{aligned} \qquad (1)$$

where $y_1(t)$ and $y_2(t)$ are unknown functions; δ is the coefficient determining friction; all coefficients are positive, excepting the coefficient α which can have any sign. In the case $\alpha > 0$ the equations (1) describe post-critical dynamics of the corresponding elastic systems. The system (1) can be obtained, in particular, in the following problems: the beam bending vibrations within framework of the Kirchhoff beam theory and the dynamics of cylindrical shells described by the Donnell equations can be considered. Then a discretization by the Bubnov-Galerkin procedure is used. If displacements of the nonlinear elastic system are approximated by a single harmonic of the Fourier series expansion for spatial coordinates, a single non-autonomous Duffing equation is obtained. Behavior of the equation was examined in numerous publications. Chaotic motions begin when the force amplitudes are slowly increased [6]. If two harmonics of the Fourier series for spatial coordinates are used, one obtains a set of two second order ODEs. Here two NNMs, which are determined by smooth trajectories in the system configuration place, exist. One of these modes can be chaotic in time in some domain of the system parameters. Thus one can formulate a problem of the stability of periodic or chaotic vibration mode in the higher-dimensional spaces. The orbital stability of trajectories of the regular or chaotic modes is determined by the numerical-analytical approach which is based on the Lyapunov definition of stability [5]. Current in time values of variations are compared with initial ones. Corresponding calculations are realized at points on some chosen mesh of the system parameter space. Calculations are conducted as long as boundaries of the stability/instability regions (in the chosen scale) on the space are variable. This is a principal criterion for the choice of the time of calculation *T*. The vibration mode $y_1 = y_1(t)$, $y_2 = 0$ is presented in Fig.2. Here a phase place of the mode is shown in Fig. 2a, and the chaotic in time behavior of the mode is observed. Phase place of variations y_2 is shown in Fig. 2b, where the variations tend to zero. So, the NNM with chaotic in time behavior is stable. Stability of standing waves in nonlinear chains can also be analyzed by the proposed approach.

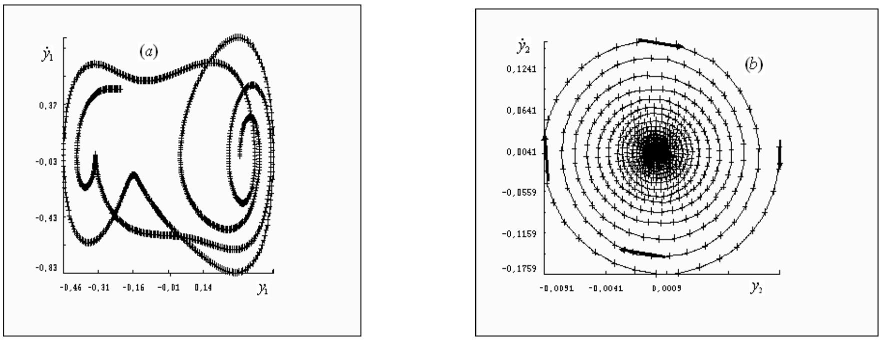

Figure 2: The stable mode of chaotic vibrations.

Conclusions

The Ince algebraization when a variable associated with the vibration mode is chosen as the new independent argument in variational equations and the test which is a consequence of the well-known Lyapunov criterion of stability, are proposed and used in analysis of stability of different stationary regimes such as nonlinear normal modes (NNMs), and traveling or standing nonlinear waves. Effectivity of these approaches is shown in some theoretical and applied problems.

References

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