PHYSICS AND MATHEMATICS

PROBABILISTIC ESTIMATION OF STABILITY OF SOLUTIONS OF OPTIMIZATION PROBLEMS

Dunaievska O. I., PhD in Technical Sciences, Associate Professor
Ukraine, Kharkiv, National technical University «Kharkiv polytechnic institute»

Abstract. *The issues of stability of solution of optimization problems are considered on the example of transportation problem of linear programming, in which the transportation costs are random variables with a known distribution density. The complexity of solving such optimization problems by classical methods is substantiated. It is proved that the problem of estimating stability admits an analytic solution if the optimal solution of the problem is sought using the matrix minimum method. This solution is based on the search for the probability that characterizes the level of stability of solving the optimization problem. The corresponding computational procedure is described. An example is considered that illustrates it graphically.*

Keywords: *Stability, optimization problem, transportation problem, matrix minimum method, random variable, distribution density, probability.*

Introduction. Stability of solutions to optimization problems is the property of the obtained solution not to change under conditions when the initial data of the problem vary in a certain range of possible values. It is accepted that the larger this range, the greater the stability of the solution. With regard to transportation problems of linear programming, the problem of estimating the stability of a solution of a problem is formulated as follows.

Let some homogeneous product be transported from m suppliers with the volumes of the delivered product $A_1, A_2, ..., A_m$ to the aggregate of consumers whose needs are equal $B_1, B_2, ..., B_n$. The costs of transportation from specific suppliers to specific consumers are random variables with known distribution costs. The system of constraints on the unknown variables of the problem that define the transportation plan, cuts out a convex hyperpolyhedron in the phase space of variables.

The solution of the problem is achieved at the extreme point of this polyhedron, in which there is a tangency of the polyhedron with the target hyperplane. The position of the hyperplane is determined by the values of transportation costs.

Since values of transportation costs are random variables, in each implementation of the set of these variables the position of the target hyperplane will be different, and the extreme value of the polyhedron in which the tangency takes place can vary.

Let a solution of the transportation problem be obtained and a particular vertex of the polyhedron corresponding to the solution is known. We construct the normal to the target hyperplane at the point of its tangency with the optimal vertex. In addition, at the same point, we construct a system of normals to all the constraints, the intersection of which forms the optimal vertex. This system of normals forms a cone inside which there is a normal to the target hyperplane (Fig. 1).

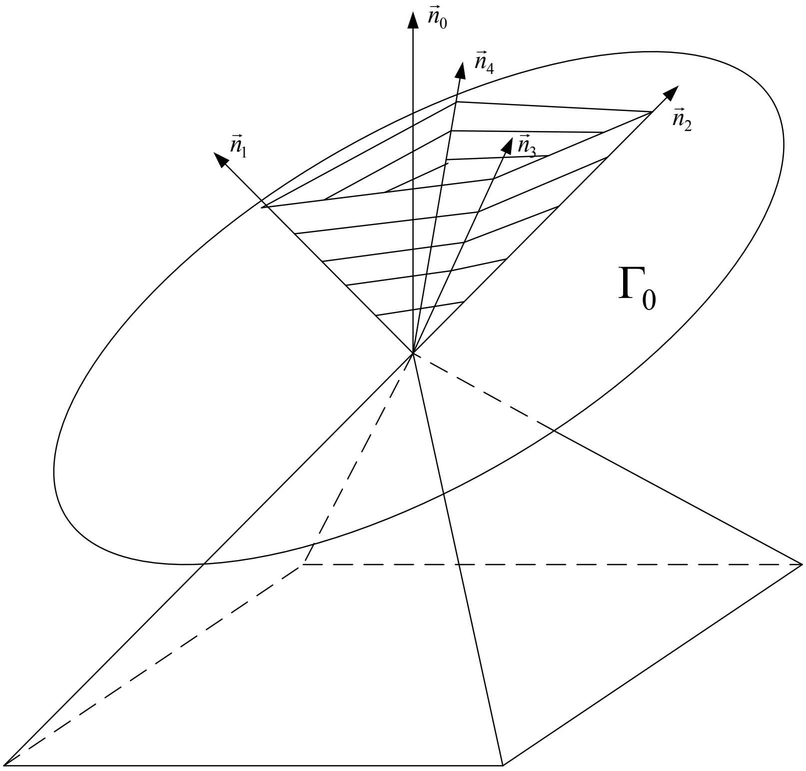

Fig. 1. The cone spanned by the system of normals to the planes – constraints

Random changes in the values of the problem cost will result in the rocking of this normal. The solution will persist until, during this rocking, the normal goes beyond the cone of constraints. In this case, the hyperplane touches the admissible vertex of the polyhedron, that is, the solution changes.

Thus, the problem consists in calculating the probability that, for known distribution densities of the random transportation costs, the normal to the target hyperplane will be located inside the cone spanned by the system of normals to the constraint hyperplanes.

Estimation of the stability of the solution of the transportation problem in such a formulation is very complicated, since an optimal vertex is sought as a result of a change of the complex multi-step algorithm. At the same time, this problem allows a simple analytic solution if the optimal solution of the transportation problem is sought using the minimal matrix method.

Results of the study. This method is structurally structured in such a way that, when used, a sequence of assignments can be unambiguously obtained, aligned in accordance with an increasing sequence of transportation costs. This sequence in the first step begins with the procedure for finding the minimum value matrix element and at each subsequent step continues the same procedure for the matrix corrected after the next step. As a result, we obtain a fixed increasing sequence of costs, which determines the order of finding the values**.** Let's renumber this sequence: $C_1 < C_2 < ... < C_{m+n+1}$.

We now calculate the probability of realizing the correct order of assignments for a certain pair $(k,\ k+1)$ of next-standing members of the sequence $C_1, C_2, ..., C_{m+n+1}$.

Let the number C_{k+1} take a random value X. We calculate the probability that a random number C_k will take a value less than X, and then integrate this probability together with the probability distribution density of the random variable C_{k+1}. It is natural to consider independent density distribution $\varphi(C_k)$ and $\varphi(C_{k+1})$ of random variables C_k and C_{k+1}. Therefore, the required probability is calculated by the formula $P_{k,k+1} = \int_0^{\infty} \varphi(C_{k+1})dC_{k+1} \int_0^{X} \varphi(C_k)dC_k$.

For example

$$\varphi(C_k)=\begin{cases}\dfrac{1}{b_k-a_k}, & C_k\in[a_k,b_k],\\ 0, & C_k\notin[a_k,b_k];\end{cases}$$

$$\varphi(C_{k+1})=\begin{cases}\dfrac{1}{b_{k+1}-a_{k+1}}, & C_{k+1}\in[a_{k+1},b_{k+1}],\\ 0, & C_{k+1}\notin[a_{k+1},b_{k+1}];\end{cases}$$

$$b_k>a_{k+1}. \tag{1}$$

Let's choose $x\in(a_{k+1},b_k)$. Then

$$P_{k,k+1}=\int_{a_{k+1}}^{b_k}\frac{dC_{k+1}}{b_k-a_{k+1}}\int_{a_k}^{\min(x,b_k)}\frac{dC_k}{b_k-a_k}=\frac{1}{(b_k-a_{k+1})(b_k-a_k)}\int_{a_{k+1}}^{b_k}(x-a_k)dC_{k+1}=$$

$$=\left|u=x-a_k\right|=\frac{1}{(b_k-a_{k+1})(b_k-a_k)}\int_{a_{k+1}-a_k}^{b_k-a_k}udC_{k+1}=\frac{u^2}{(b_k-a_{k+1})(b_k-a_k)2}\Bigg|_{a_{k+1}-a_k}^{b_k-a_k}=$$

$$=\frac{(b_k-a_a)^2-(a_{k+1}-a_k)^2}{2(b_k-a_{k+1})(b_k-a_k)}=\frac{(b_k-a_k+a_{k+1}-a_k)(b_k-a_k-a_{k+1}+a_k)}{2(b_k-a_{k+1})(b_k-a_k)}=$$

$$=\frac{(b_k+a_{k+1}-2a_k)(b_k-a_{k+1})}{2(b_k-a_{k+1})(b_k-a_k)}=\frac{b_k+a_{k+1}-2a_k}{2(b_k-a_k)}. \tag{2}$$

In this case, the probability of realizing the correct order of assignments in the algorithm of the minimal matrix element is

$$P=\prod_{k=1}^{m+n-2}P_{k,k+1}=\frac{1}{2^{m+n-2}}\prod_{k=1}^{m+n-2}\frac{b_k+a_{k+1}-2a_k}{b_k-a_k}. \tag{3}$$

The calculated value P characterizes the level of stability of the solution of the transportation problem. The chosen solution will be absolutely stable if for all pairs $(k,\, k+1)$ the inequation (1) will not be fulfilled, that is, will take place $b_k<a_{k+1}$. In this case $x>b_k$ and the inner integral in (2) will be equal to 1. In this case, of course, we get $P_{k,k+1}=1$ and $P=\prod_{k=1}^{m+n-2}P_{k,k+1}=1$.

In the other extreme case, when $a_1=a_2=\ldots=a_{m+n-1}$ and $b_1=b_2=\ldots=b_{m+n-1}$, we get $P_{k,k+1}=\dfrac{1}{2}$ and $P=\dfrac{1}{2^{m+n-1}}$.

Let's consider an example. Let the average costs of transportation from a particular supplier to a specific consumer of a certain homogeneous product are $C_1^{(0)}=2$, $C_2^{(0)}=4$, $C_3^{(0)}=6$, $C_4^{(0)}=7$.

And let the corresponding distribution densities of the random transportation costs look like:

$$\varphi(C_1)=\begin{cases}\dfrac{1}{4}, & C_1\in[0,4],\\ 0, & C_1\notin[0,4];\end{cases},\quad \varphi(C_2)=\begin{cases}\dfrac{1}{2}, & C_2\in[3,5],\\ 0, & C_2\notin[3,5];\end{cases},$$

$$\varphi(C_3)=\begin{cases}\dfrac{1}{4}, & C_3\in[4,8],\\ 0, & C_3\notin[4,8];\end{cases},\quad \varphi(C_4)=\begin{cases}\dfrac{1}{5}, & C_4\in[5.5,8.5],\\ 0, & C_4\notin[5.5,8.5];\end{cases}$$

Transportation Cost Matrix

$$C=\begin{array}{|c|c|c|l}
\hline
8 & \overset{1}{2} & \overset{4}{7} & 20 \\
\hline
\overset{2}{4} & 10 & \overset{3}{6} & 30 \\
\hline
\multicolumn{1}{c}{16} & \multicolumn{1}{c}{10} & \multicolumn{1}{c}{24} &
\end{array}$$

As a result of applying the minimal matrix method, we obtain the following transportation plan:

0	10	10	20
16	0	14	30
16	10	24	

For each pair C_k calculate the probabilities according to the formula (2):

$$P_{1,2}=\frac{b_1+a_2-2a_1}{2(b_1-a_1)}=\frac{4+3}{4\cdot 2}=\frac{7}{8};$$

$$P_{2,3}=\frac{b_2+a_3-2a_2}{2(b_2-a_2)}=\frac{4+5-6}{2\cdot(5-3)}=\frac{3}{4};$$

$$P_{3,4}=\frac{b_3+a_4-2a_3}{2(b_3-a_3)}=\frac{(8+5.5-2\cdot 4)}{2\cdot(8-4)}=\frac{5.5}{8}\cong 0.7;$$

In this case, the probability of realizing the correct order of assignments in the algorithm of the minimal matrix element is:

$$P=P_{1,2}\cdot P_{2,3}\cdot P_{2,4}=\frac{7}{8}\cdot\frac{3}{4}\cdot 0.7\cong 0.47$$

The obtained probability characterizes the stability level of the solution of the transportation problem.

Conclusions. The issues of stability of the solution of optimization problems are considered on the example of the transportation problem of linear programming, in which the transportation costs are random variables with a known distribution density. The criterion of stability is developed - the probability of realizing the correct order of assignments in the algorithm of the minimal matrix element.

REFERENCES

1. Raskin L.G. Analysis of complex systems and elements of optimal control theory / L.G. Raskin. – M.: Soviet Radio, 1976. – 344p.

2. Orlovskyy S.A. Decision-making problems with fuzzy source information. S.A.Orlovskyy– M.: Nauka (Science), 1981. – 264p.

3. Negoitse K. Application of the theory of systems to control problems: Translation from English. K.Negoitse –M.: MIR (World), 1981. – 219p.

4. Dunaievska O.I. Getting the initial support plan of the multi-index transport logistics task. / Akhiezer E.B., Geliarovska O.A., Protsai N.T. // Radioelectronics and Informatics. – 2014. – №2. – p.16-18.

5. Dunaievska O.I. Calculation of the cost of optimal routes for the aggregate of pairs (supplier - consumer) // Materials of the X International scientific and practical conference «Trends of modern science», May 30 - June 7, 2014. – Sheffield. - Science and education LTD – 2014. - p. 17-20

6. Dunaievska O.I. The task of transport logistics with a random cost of transportation / L.G.raskin, O.I. Dunaievska // Journal of Kharkiv National Automobile and Highway University. – 2007. - №37.