

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
NATIONAL TECHNICAL UNIVERSITY
“KHARKIV POLYTECHNIC INSTITUTE”

SELF-LEARNING GUIDE

“MATHEMATICS IN QUESTIONS AND ANSWERS.

Part 2”

for international students

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PREFACE

This methodic guide is designed for international students who study at the preparatory departments of Ukrainians universities. In this methodic guide students can find the basic theoretical questions and answers for the main topics to be covered during the Math educational course (2 semester).

The special attention to the basic concepts of the Mathematics theory and scientific style in Mathematics are focused in the methodic guide.

The methodic guide includes some basic problems of Function, Inequalities and Trigonometry.

Active learning style encouraged throughout, helping students really get to grip with the theory and getting them ready for successful mastery of higher Mathematics course.

This methodic guide may be useful for international students of preparatory courses at universities, secondary schools, high schools and colleges. The book is also intended to be useful for Mathematics teachers who work with international students.

Authors believe that this methodic guide should not just be a collection of mathematical facts carefully laid out, so the student can open it and cram in whatever formula they have to remember for tomorrow's exam.

A Math methodic guide, especially for international students, should be full of questions, not just exercises. Questions that require some thought to answer. After all, "it's the questions that keep the students interested, not the answers." The basic issues of the theory give the possibility to address gaps in knowledge, acquire the knowledge needed for all major sections from the school Mathematics and revise the theory independently, without any other assistance.

Topic 1. FUNCTION

1. Write a definition of function?

Answer. A function is a relation between a set of inputs X and a set of permissible outputs Y with the property that each input is related to exactly one output. It is denoted as follows: $y=f(x)$.

2. What is an independent variable?

Answer. Variable " x " is called the independent variable or argument if $y=f(x)$.

3. What is a dependent variable?

Answer. Variable " y " is called the dependent variable or function if $y=f(x)$.

4. What does the symbol f denote if $y=f(x)$?

Answer. The symbol f is the law which denotes that each " x " value of function is related to its " y " value.

5. What is a domain of definition of function?

Answer. The domain of definition of function is all the possible input values " x " for which the function is defined. Domain of definition of function is denoted by $D(f)$.

6. What is a range of definition of function?

Answer. The range of definition of function ($E(f)$) is the set of all possible output values (usually the variable " y ", or sometimes expressed as $f(x)$), which result from using a particular function.

7. What three basic methods for determining functions do you know?

Answer. There are three basic methods for determining functions: analytical, tabular and graphical.

8. What do you know about analytical method for determining functions?

Answer. The analytical method consists in the fact that the function is imposed by a formula, for example $y = x^2$, $y = 8mx$, $y = 5\sin 3x$.

9. What do you know about tabular method for determining functions?

Answer. The tabular method consists in the fact that the function is imposed in tabular form. For example, there are tables of numbers of cubes, squares of numbers, trigonometric functions, logarithmic functions etc.

10. What do you know about graphical method for determining functions?

Answer. Graphical method consists in the fact that the graph of the $y=f(x)$ function shows in the x, y -Plane (or “Cartesian” plane).

11. What is a graph of function $y=f(x)$?

Answer. Graph of the function $y=f(x)$ is the collection of all ordered pairs $(x, f(x))$ (Fig. 1).

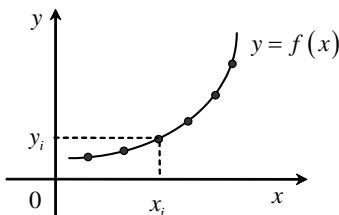


Figure 1

12. What is increasing function?

Answer. Monotonically increasing (or non-decreasing) on the (a, b) if their values are only rising and never falling with increasing values of x ($f(x_1) \leq f(x_2)$ or $f(x_1) < f(x_2)$ with $x_1 < x_2$) (Fig. 2 a, b).

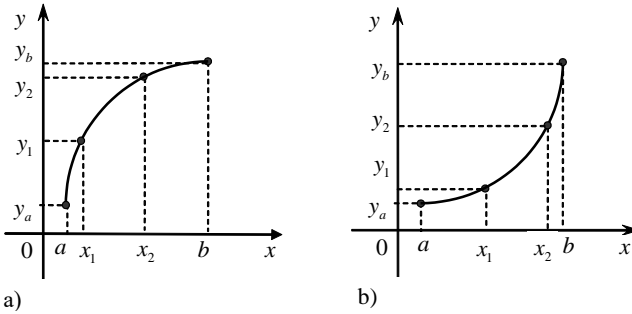


Figure 2

13. What is decreasing function?

Answer. Monotonically decreasing (or non-increasing) functions on the (a, b) if their values are only falling and never rising ($f(x_1) \geq f(x_2)$ or $f(x_1) > f(x_2)$ with $x_1 < x_2$) (Fig. 3 a, b).

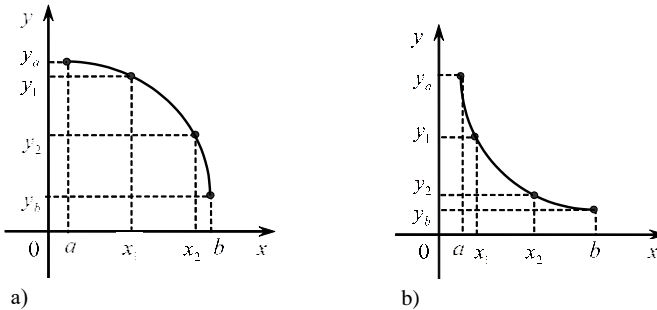


Figure 3

14. What functions are even ones?

Answer. A function $y = f(x)$, defined on a symmetric set $D(f)$, is called even one, if for $x \in D(f)$ the equality $f(-x) = f(x)$ is performed (Fig. 4).

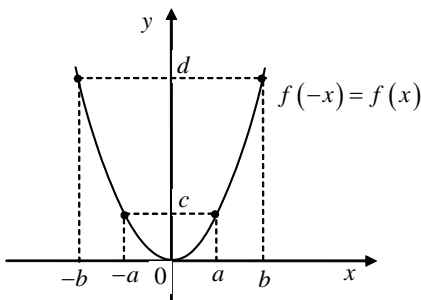


Figure 4

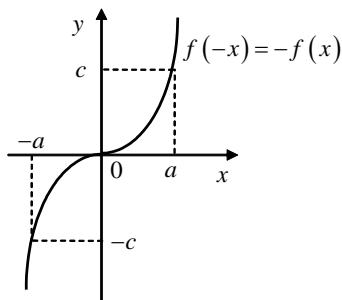


Figure 5

15. What functions are odd ones?

Answer. A function $y = f(x)$ defined on a symmetric $D(f)$ set is called an odd function, if for $x \in D(f)$ the equality $f(-x) = -f(x)$ is performed (Fig. 5).

16. What functions are neither even nor odd ones?

Answer. For example, $y = x + 2$ is neither even nor odd function because $f(-x) \neq \pm f(x)$, $y = \ln x$ is neither even nor odd function because the domain of its definition is asymmetric ($D(f) = \mathbb{R}^+$).

17. What functions are periodic ones?

Answer. The function $y = f(x)$ is called periodic one if there is a number $T \neq 0$ that the numbers $(x - T)$ and $(x + T)$ also belong to the domain of definition for any x and the following equality $f(x - T) = f(x + T) = f(x)$ is performed. The number T is called the period of $y = f(x)$ function. For example, $y = \sin x$ is a periodic function with period $T = 2\pi$.

18. What do you know about “zeros” (roots) of a function?

Answer. “Zeros” (roots) of a function is an input value $x \in D(f)$ that produces an output of zero $y = f(x) = 0$. For example, for the function $y = \sin x$ the zeros are the following points: $x = k\pi$, $k \in \mathbb{Z}$.

19. What is an asymptote of a function?

Answer. Asymptote of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity. Asymptotes may be: vertical ($x=a$), horizontal ($y=b$) and slope ($y=kx+b$) (Fig. 6, 7).

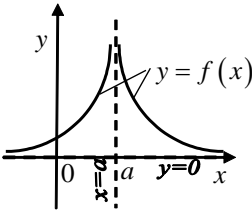


Figure 6

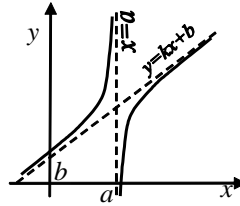


Figure 7

20. What is an asymptote of a function?

Answer. The basic properties of function

1. The domain of definition of the function $D(f)$.
2. The range of definition of the function $E(f)$.
3. Even, odd or neither function.
4. Periodic or aperiodic function.
5. The zeros (roots) of function.
6. Monotonicity of function (intervals of increasing or decreasing).
7. Asymptotes of function.

21. What is an inverse function?

Answer. Inverse function is a function that reverses to another function: if the function $y = f(x)$ applied to an input x gives a result of y , then applying the inverse function φ to y gives the result x , and vice versa, e.g. $f(x) = y$ and $\varphi(y) = x$. An even function does not have the inverse function.

22. What do you know about the domain and range of an inverse function?

Answer. The domain of an inverse function coincides with the range of definition of the original function ($D(\varphi) = E(f)$), and range of definition of an inverse function coincides with the domain of the original function ($E(\varphi) = D(f)$). For example, for the function $y = x^3$ ($D(f) = R$ and $E(f) = R$) the inverse function $y = \sqrt[3]{x}$ can be determined from the equality $D(\varphi) = R$ or $E(\varphi) = R$.

23. What do you know about the domain and range of an inverse function?

Answer. Comparing graphs, we can make conclusion: graphics of inverse functions ($y = x^3$ (curve I) and $y = \sqrt[3]{x}$ (curve II)) are symmetrical with respect to $y = x$ (Fig. 8).

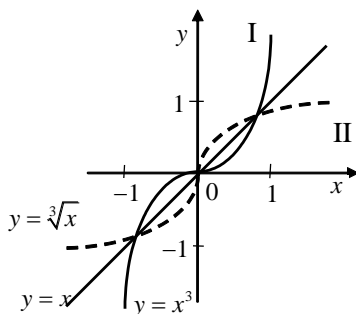


Figure 8

24. What is a linear function?

Answer. The formula of a linear function. function is: $y = kx + b$, where k and b are constants. The graph of a linear function is a straight line (Fig. 9, 10).

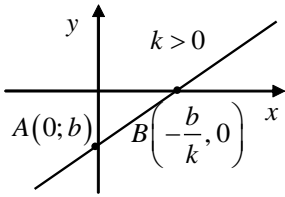


Figure 9

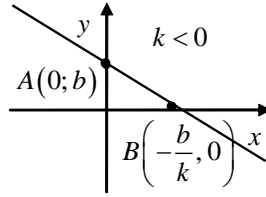


Figure 10

25. What properties of a linear function do you know?

Answer. Properties of linear function $y = kx + b$.

- 1) The domain of the linear function is the real axis: $D(f) = R$.
- 2) The range of this function: $E(f) = (-\infty, +\infty)$.
- 3) The function increases monotonously, if $k > 0$ (Fig. 9) and decreases monotonously, if $k < 0$ (Fig. 10).
- 4) Zeros of the function: $y = 0$, if $x = -\frac{b}{k}$.
- 5) If $k > 0$, then $y > 0$ with $x > -\frac{b}{k}$, $y < 0$ with $x < -\frac{b}{k}$.
If $k < 0$, then $y > 0$ with $x < -\frac{b}{k}$, $y < 0$ with $x > -\frac{b}{k}$.
- 6) The function is neither even, nor odd.
- 7) The function has no extremum.
- 8) Graph of the function has no asymptotes.

26. What is a power function?

Answer. The function $y = kx^n$ ($k, n \in R$) and x is variable is called a power function.

If $n = 1$, we have a direct proportionality $y = kx$ (k is a coefficient of direct proportionality). If $n = -1$, we have an inverse proportionality

$$y = \frac{k_1}{x} \quad (k_1 \text{ is a coefficient of inverse proportionality}).$$

27. What is a graph of power function?

Answer. If $n=2$ in formula $y = x^n$, then the graph of function is a parabola $y = x^2$ (Fig. 11).

If $n=3$, then the graph of function is cubic parabola $y = x^3$ (Fig. 12).

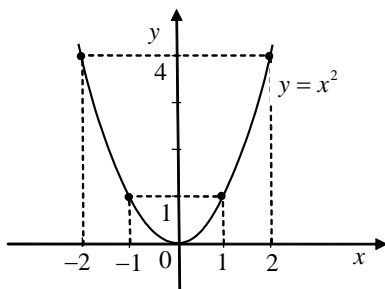


Figure 11

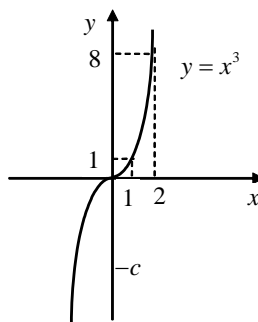


Figure 12

28. What properties of a power function do you know?

Answer.

Properties of function $y = x^2$

1. $D(f) = R$.
2. $E(f) = R^+ = (0, \infty)$,
 y is a positive number.
3. Function has one zero
($y = 0$, if $x = 0$).
4. $y > 0$, if
 $x \in (-\infty, 0) \cup (0, +\infty)$.
5. Function decreases on the interval
 $x \in (-\infty, 0)$ and increases on
the interval $x \in (0, +\infty)$.

Properties of function $y = x^3$

1. $D(f) = R$.
2. $E(f) = R$.
3. Function has one zero
($y = 0$, if $x = 0$).
4. $y > 0$, if $x \in (0, +\infty)$ and
 $y < 0$, if $x \in (-\infty, 0)$
5. Function increases
monotonously at all domain of
function.

6. Function has minimum, if $x=0$, $y_{\min}(0)=0$.
7. Function is even: $y(-x)=y(x)$. The graph of function is symmetric with respect to y -axis.
8. The graph of function has not asymptotes.
6. Function has not extrema.
7. Function is odd: $y(-x)=-y(x)$. The graph of function is symmetric with respect to the origin.
8. The graph of function has not asymptotes.

29. What is a quadratic function?

Answer. Function $y=ax^2+bx+c$, namely, a quadratic function, obtained from a polynomial of n degree, if $n=2$. The domain of the function $y=ax^2+bx+c$ is: $D(y)=R$.

30. What is a graph of quadratic function?

Answer. The graph of the function is a parabola. The graph of a parabola either opens upward like $y=x^2$, if $a>0$ or opens downward like the graph of $y=-x^2$, if $a<0$.

31. What coordinates of vertex of the parabola do you know?

Answer. Coordinates of vertex of the parabola are given with formulas: $x_0=-\frac{b}{2a}$, $y_0=\frac{4ac-b^2}{4a}$.

32. What fractional rational function do you know?

Answer. Fractional rational function is the function of the form:

$$y=\frac{k}{x} \text{ (inverse proportionality).}$$

If $k=1$ the function $y=\frac{k}{x}$ has the form $y=\frac{1}{x}$ (Fig.13). If $k=-1$ it

has the form $y=-\frac{1}{x}$ (Fig.14).

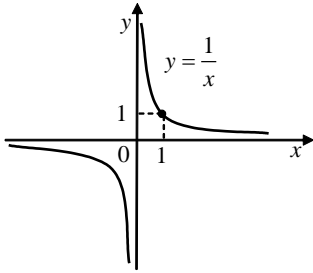


Figure 13

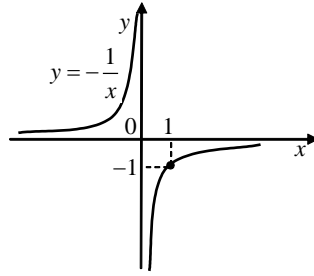


Figure 14

33. What is a graph of function $y = \frac{k}{x}$?

Answer. The graph of function $y = \frac{k}{x}$ is hyperbola.

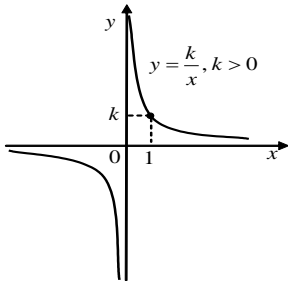


Figure 15

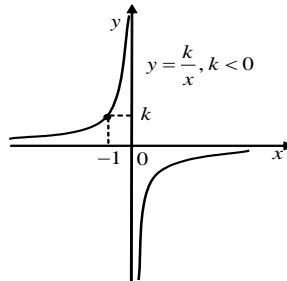


Figure 16

34. What is irrational function?

Answer. Function of the form $y = \sqrt[n]{x}$ is called an irrational function.

35. Draw the graphs of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$ functions?

Answer.

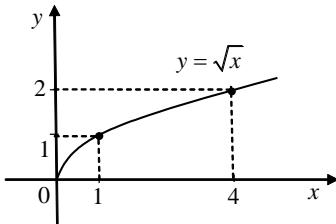


Figure 17

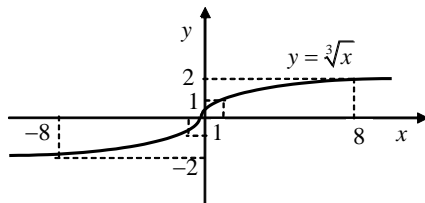


Figure 18

36. What is absolute value function $y = |f(x)|$?

Answer. This function $y = |x|$ is denoted by formula:

$$|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

For the function $y = |f(x)|$ it can be written down:

$$|f(x)| = \begin{cases} f(x), & \text{if } f(x) \geq 0, \\ -f(x), & \text{if } f(x) < 0. \end{cases}$$

37. Draw the graphs of function $y = |x|$?

Answer.

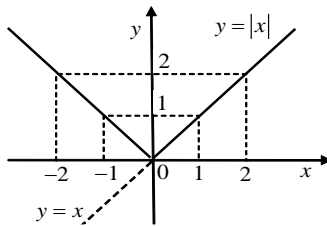


Figure 19

38. What is exponential function?

Answer. Function, given by the formula $y = a^x$, where $a > 0$, $a \neq 1$ is called exponential.

39. What is a graph of exponential function?

Answer.

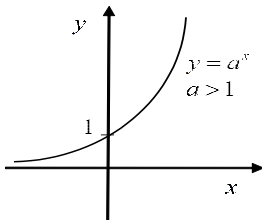


Figure 20

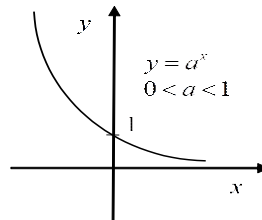


Figure 21

40. What is logarithmic function?

Answer. Function, given by formula $y = \log_a x$ ($a > 0$, $a \neq 1$) is called logarithmic.

41. What is a graph of logarithmic function?

Answer. The logarithmic function $y = \log_a x$ is the inverse one for the exponential function $y = a^x$. Graphs of these functions are symmetric with respect to the line $y = x$ (Fig. 22, 23).

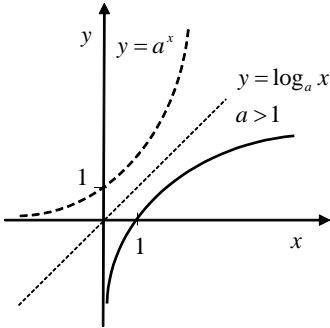


Figure 22

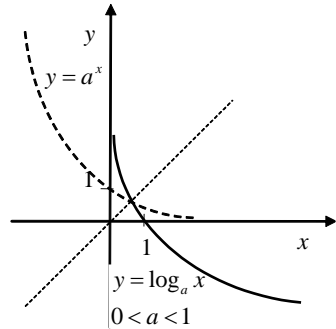


Figure 23

42. What Basic Trigonometric Functions do you know?

Answer. The basic trigonometric functions are: $y = \sin x$; $y = \cos x$; $y = \tan x$; $y = \cot x$.

43. What is a graph of function $y = \sin x$; ?

Answer. The graph of function $y = \sin x$; is called sinusoid.

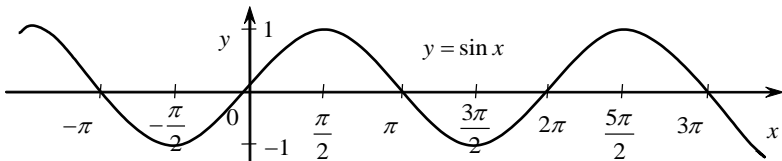


Figure 24

44. What is a graph of function $y = \cos x$; ?

Answer. The graph of function $y = \cos x$; is called the cosine curve.

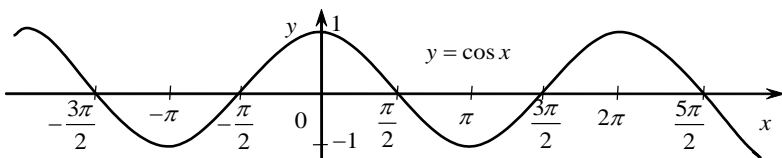


Figure 25

45. What is a graph of function $y = \tan x$?

Answer. Graph of the function $y = \tan x$; is called the tangent curve.

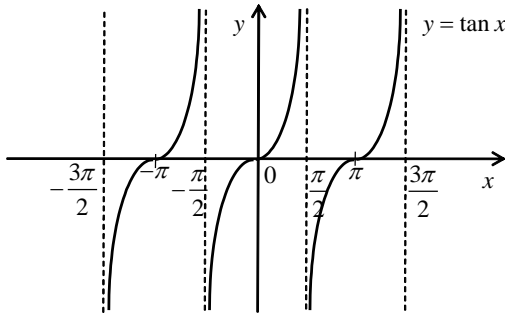


Figure 26

46. What is a graph of function $y = \cot x$?

Answer. Graph of function $y = \cot x$ is called the cotangensoid.

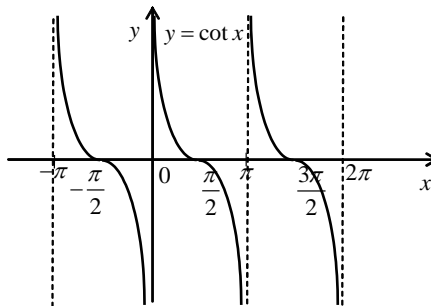


Figure 27

47. What Inverse Trigonometric Functions?

Answer. Functions inversed to the functions $y = \sin x$; $y = \cos x$; $y = \tan x$; $y = \cot x$ on appropriate intervals, are called the inverse trigonometric functions. They are: $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$, $y = \text{arccot } x$.

Topic 2. TRANSFORMATIONS OF FUNCTIONS

48. How could you obtain a graph of the function $y = A \cdot f(x)$?

Answer. Graph of the function $y = A \cdot f(x)$ is obtained with the graph $y = f(x)$. If $A > 1$, the graph is stretched vertically and if $0 < A < 1$, the graph is shrunk vertically (Fig. 28).

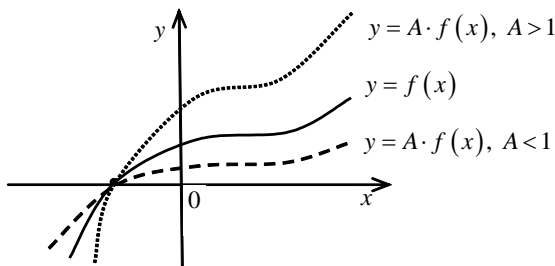


Figure 28

49. How could you obtain a graph of the function $y = f(\alpha \cdot x)$?

Answer. Graph of the function $y = f(\alpha \cdot x)$ is obtained from the graph $y = f(x)$ with its stretching or shrinking along the x -axis horizontally. If $|\alpha| > 1$, the graph is stretched horizontally and if $0 < \alpha < 1$, the graph is shrunk horizontally.

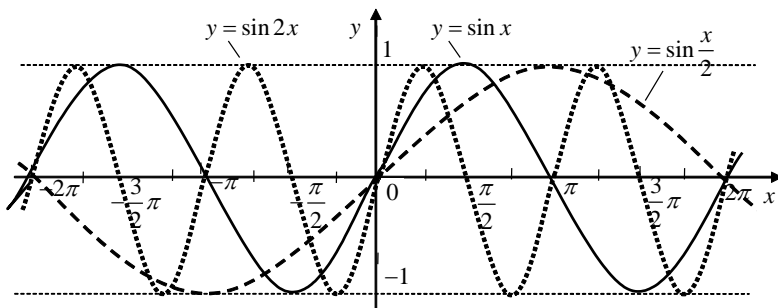


Figure 29

50. How could you obtain a graph of the function $y=f(x+b)$?

Answer. The graph of function $y=f(x+b)$ is obtained with the graph $y=f(x)$ moving by the value $x=|b|$ to the left along the x -axis (in the negative direction of the x -axis, if $b>0$), and to the right (in the positive direction of the x -axis if $b<0$).

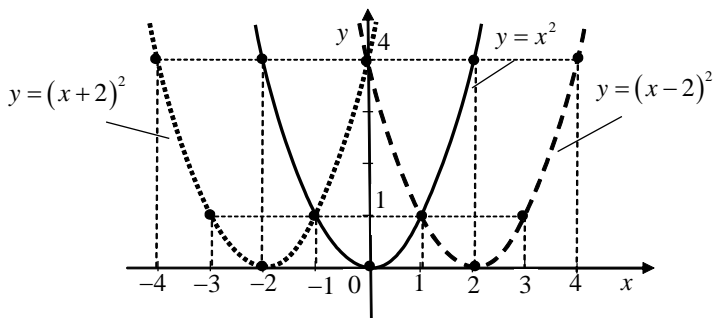


Figure 30

51. How could you obtain a graph of the function $y=f(x)+M$?

Answer. Graph of function $y=f(x)+M$ is obtained with the graph $y=f(x)$ by vertical translation at the value $y=|M|$ upward, if $M>0$ and downward, if $M<0$.

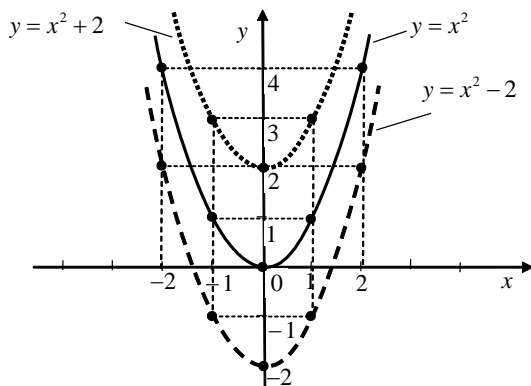


Figure 31

52. How could you obtain a graph of the function $y = -f(x)$?

Answer. The graph of function $y = -f(x)$ is obtained with reflecting the graph of function $y = f(x)$ across the x -axis.

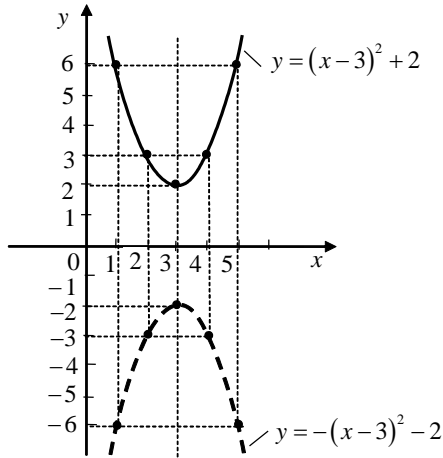


Figure 32

53. How could you obtain a graph of the function $y = f(-x)$?

Answer. Graph of function $y = f(-x)$ is obtained with reflection the graph of function $y = f(x)$ across the y -axis.

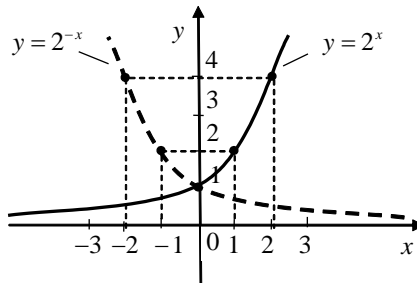


Figure 33

54. How could you obtain a graph of the function $y = |f(x)|$?

Answer. Graph of function $y = |f(x)|$ is obtained from the graph of function $y = f(x)$ with reflection across the x -axis. We make the reflecting the part of graph below the x -axis, $f(x) < 0$. The part of graph above the x -axis ($f(x) > 0$) is without changing.

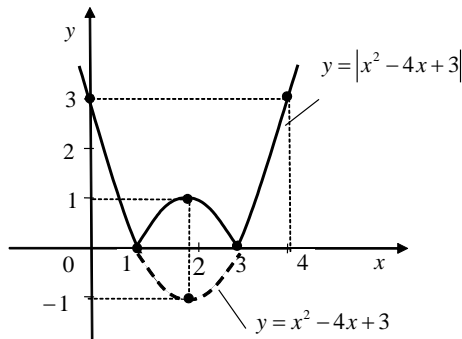


Figure 34

55. How could you obtain a graph of the function $y = f(|x|)$?

Answer. Graph of function $y = f(|x|)$ is obtained from the graph $y = f(x)$: the graph of function $y = f(x)$ is saved only if $x \geq 0$, and it reflects symmetrically across the y -axis.

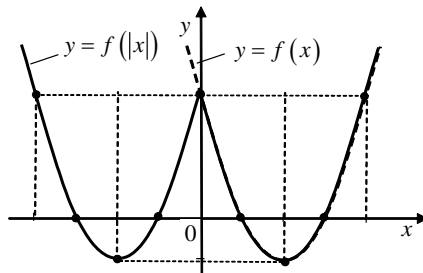


Figure 35

Topic 3. RATIONAL, EXPONENTIAL AND LOGARITHMIC INEQUALITIES

56. What is an inequality?

Answer. An inequality is a relation that holds between two values when they are different. If two expressions are not equal, they can contain following signs: “greater than” ($>$), “less than” ($<$), “greater than or equal to” (\geq), “less than or equal to” (\leq).

$f(x) > g(x)$, $f(x) < g(x)$, $f(x) \geq g(x)$, $f(x) \leq g(x)$ are called inequalities.

57. What is a solution of the inequality?

Answer. Solution of inequality is the value of a variable that makes an inequality true.

58. What does it mean to solve an inequality?

Answer. To solve an inequality means to find all its solutions or to prove that there are not any ones.

59. Which inequalities are called as equivalent ones?

Answer. Two inequalities with one variable are called the equivalent ones, if the solutions of these inequalities coincide.

60. What theorems for solving inequalities do you know?

Answer. We know following theorems.

Theorem 1. We may add any positive or negative number to both sides of an inequality and obtain an equivalent inequality to the original one (the sense will not change).

$$P(x) > Q(x) \Leftrightarrow P(x) \pm T(x) > Q(x) \pm T(x).$$

For example, $x^2 + 7x < x^2 \Leftrightarrow 7x < 0 \Leftrightarrow x < 0$.

Theorem 2. We may move any term of inequality to another side with the opposite sign and obtain an equivalent inequality to the original one (the sense will not change).

$$P(x)+T(x)>Q(x)\Leftrightarrow P(x)>Q(x)-T(x).$$

For example, $x^2 - 4x > -3 \Leftrightarrow x^2 - 4x + 3 > 0$

Theorem 3. We may multiply or divide both sides of an inequality by any positive number and obtain an inequality equivalent to the original one.

$$P(x)>Q(x), T(x)>0, x \in R \Leftrightarrow P(x) \cdot T(x) > Q(x) \cdot T(x)$$

For example, $\frac{1}{6}x \geq -5 \Leftrightarrow x \geq -30$.

Theorem 4. If we multiply or divide both sides of an inequality by a negative number, we should reverse the direction of the inequality sign.

$$P(x)>Q(x); T(x)<0; x \in R \Leftrightarrow P(x) \cdot T(x) < Q(x) \cdot T(x).$$

For example, $-7x \leq 21 \Leftrightarrow x \geq -3$.

61. Write the standard forms of linear inequalities with a single variable?

Answer. $ax > b$, $ax < b$, $ax \leq b$, $ax \geq b$ are linear inequalities with a single variable.

62. Write the standard forms of linear inequalities with a single variable?

Answer. Inequalities $ax^2 + bx + c > 0$, $ax^2 + bx + c \geq 0$, $ax^2 + bx + c < 0$ or $ax^2 + bx + c \leq 0$, where $a \neq 0$ are called quadratic inequalities.

63. What method do we often use for solving rational inequalities?

Answer. We often solve rational inequalities with method of intervals.

64. How many inequalities may system of inequalities with one variable consist of?

Answer. The system of inequalities with one variable may consist of two or more inequalities.

65. What does it mean to solve the system of inequalities mean?

Answer. To solve the system of inequalities means to find the set of all solutions or to prove that there are no solutions. To solve the system of inequalities with one variable, we need to solve each inequality separately and find the intersection of solutions.

66. How do we solve absolute value inequalities?

Answer. To solve absolute value inequalities, we need to consider the definition of absolute value function:

$$|f(x)| = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ -f(x), & \text{if } f(x) < 0. \end{cases}$$

Inequality with absolute value is equivalent to two systems of inequalities that can be written as intersection or union of two inequalities.

$$1. \text{ If } |f(x)| > g(x), \text{ then } \begin{cases} f(x) \geq 0 \\ f(x) > g(x) \\ f(x) < 0 \\ -f(x) > g(x) \end{cases} \Leftrightarrow \begin{cases} f(x) > g(x) \\ f(x) < -g(x) \end{cases}.$$

$$2. \text{ If } |f(x)| < g(x), \text{ then } \begin{cases} f(x) \geq 0 \\ f(x) < g(x) \\ f(x) < 0 \\ -f(x) < g(x) \end{cases} \Leftrightarrow \begin{cases} f(x) < g(x) \\ f(x) > -g(x) \end{cases}.$$

We use the method of intervals for solving absolute value inequalities.

67. Which inequalities are called as irrational inequalities?

Answer. Inequalities that contain a variable under the radical are called irrational inequalities. For example, $\sqrt{x+5} > x$, $\sqrt[3]{x^2+5} < x-7$ are irrational inequalities.

68. How do we solve irrational inequalities?

Answer. There are two possible cases in solving irrational inequalities.

$$1. \sqrt{P(x)} < Q(x) \Leftrightarrow \begin{cases} P(x) \geq 0 \\ P(x) < [Q(x)]^2 \\ Q(x) > 0 \end{cases}.$$

The solution of this system is the solution of original inequality.

$$2. \sqrt{P(x)} > Q(x) \Leftrightarrow \begin{cases} P(x) \geq 0 \\ Q(x) \geq 0 \\ (P(x))^2 > (Q(x))^2 \\ P(x) \geq 0 \\ Q(x) < 0 \end{cases}.$$

69. What is logarithm?

Answer. The logarithm of a number x with respect to base b is the exponent by which b must be raised to yield x . In other words, the logarithm of x to base b is the solution y to the equation $b^y = x$.

70. What logarithmic identities do you know?

Answer. Logarithmic identities are follows:

- $b^{\log_b x} = x$, because $\text{antilog}_b(\log_b x) = x$.
- $\log_b b^x = x$, because $\log_b(\text{antilog}_b x) = x$.
- common logarithm: $\log_{10} b = \lg b$.
- natural logarithm: $\log_e b = \ln b$, ($e = 2.718281828\dots$ is the base of natural logarithms).

71. What basic properties of logs (Algebraic Identities or Laws) do you know?

Answer. Basic Properties of Logs (Algebraic Identities or Laws):

- 1) $a^{\log_a b} = b$ (Basic logarithmic identity or reciprocal property);
- 2) $\log_a a = 1$ (Trivial identity or reciprocal property);
- 3) $\log_a 1 = 0$ (Trivial identity or reciprocal property);

- 4) $\log_a bc = \log_a b + \log_a c$, if $b > 0$, $c > 0$ (The log of a product is equal to the sum of the logs of the factors);
- 5) $\log_a \frac{b}{c} = \log_a b - \log_a c$, if $b > 0$, $c > 0$ (The log of a quotient is equal to the difference between the logs of the numerator and denominator);
- 6) $\log_a b^p = p \cdot \log_a b$, if $b > 0$ (The log of a power is equal to the power times the log of the base);
- 7) $\log_a b = \frac{\log_c b}{\log_c a}$, if $b > 0$ (Changing the base);
- if $c = b$, then $\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$;
- 8) $\log_a b = \log_{a^p} b^p$ (Using simpler operations);
- 9) $\log_a \sqrt[k]{m} = \log_a m^{\frac{1}{k}} = \frac{1}{k} \cdot \log_a m$ (Using simpler operations);
- 10) $\log_a b = -\log_a \frac{1}{b}$ (Using simpler operations);
- 11) $a^{\log_c b} = b^{\log_c a}$.

72. What is exponential equation?

Answer. Exponential equations are the equations in which variables occur in the exponents. For example, $(f(x))^{\varphi(x)} = (f(x))^{g(x)}$, $2^x = 3^{x-1}$, $5^{x^2-8} - 2 = 0$, $a^{f(x)} = a^{\varphi(x)}$ are exponential equations.

73. What Techniques for Solving Exponential Equations do you know?

Answer. We know following Techniques for Solving Exponential Equations:

- I. Same Base Problems.** $a^{f(x)} = a^{\varphi(x)}$, $a > 0$, $a \neq 1$ or $a^{f(x)} = 1$, $a > 0$, $a \neq 1$. If $a^0 = 1$, and if $a^{f(x)} = a^0$, it means $f(x) = 0$.

II. Dividing Both Sides of the Equation $a^{f(x)} = b^{f(x)}$, $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$ by $a^{f(x)}$ or $b^{f(x)}$.

III. Making the Same Base of the Logarithms of Both Sides of the Equation $A \cdot a^{f(x)} = B \cdot b^{g(x)}$, $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$.

IV. Factoring Exponential Equations

V. Evaluating to the Quadratic Equation. Evaluate the form $A \cdot a^{2f(x)} + B \cdot a^{f(x)} + C = 0$ ($a > 0$, $a \neq 1$) to a quadratic equation $Ay^2 + By + C = 0$ with substitution $a^{f(x)} = y$.

VI. Inserting a New Variable

74. What is a logarithmic equation?

Answer. The equation is called *logarithmic* if it contains logarithmic expressions and constants. For example, the equations: $\log_a f(x) = \log_a g(x)$, $\log_x 4 = -2$, $\log_3 x = 2$ are logarithmic equations.

75. What is a trivial logarithmic equation?

Answer. Trivial logarithmic equations are the equations of the form:

- a) $\log_a x = b$; b) $\log_a f(x) = b$; c) $\log_a f(x) = g(x)$;
d) $\log_a f(x) = \log_a g(x)$; e) $\log_{\varphi(x)} f(x) = \log_{\varphi(x)} g(x)$.

76. Which techniques for solving trivial logarithmic equations do you know?

Answer. It is used different techniques for solving trivial logarithmic equations.

1. Solving Logarithmic Equations Using Some Logarithm Properties.

- a) $\log_a x = b \Leftrightarrow x = a^b$, $a > 0$, $a \neq 1$.
b) $\log_a f(x) = b \Leftrightarrow f(x) = a^b$, $a > 0$, $a \neq 1$.
c) $\log_a f(x) = g(x) \Leftrightarrow f(x) = a^{g(x)}$, $a > 0$, $a \neq 1$.

$$d) \log_a f(x) = \log_a g(x) \Leftrightarrow \begin{cases} f(x) = g(x) \\ f(x) > 0 \\ g(x) > 0 \end{cases}, \text{ where } a > 0, a \neq 1.$$

$$e) \log_{\varphi(x)} f(x) = \log_{\varphi(x)} g(x) \Leftrightarrow \begin{cases} f(x) = g(x) \\ f(x) > 0 \\ g(x) > 0 \\ \varphi(x) > 0 \\ \varphi(x) \neq 1 \end{cases}.$$

2. Solving Logarithmic Equations Using Log Rules.
3. Solving Equations Using the Basic Logarithmic Identity $a^{\log_a b} = b$.
4. Solving Logarithmic Equations with Inserting New Variable (Substitution).
5. Solving Equations Using Logarithms.
6. Solving Equations by Dividing Both Sides of the Equation.
7. Solving Equations Using the Formulas for Another Base.

77. What is an exponential inequality?

Answer. An inequality with variables in the exponent ($a^{f(x)} > b^{g(x)}$) is called an Exponential Inequality.

78. Which techniques for solving exponential inequalities do you know?

Answer. In the process of solving exponential inequality it can be reasonable to make the same base of both sides of the inequality: $a^{f(x)} > a^{\varphi(x)}$. There can be two cases:

- a) $a > 1, a^{f(x)} > a^{\varphi(x)} \Rightarrow f(x) > \varphi(x)$;
- b) $0 < a < 1, a^{f(x)} > a^{\varphi(x)} \Rightarrow f(x) < \varphi(x)$.

For solving such inequalities as $u(x)^{f(x)} > u(x)^{\varphi(x)}$, it is necessary to solve the systems of inequalities.

There are two cases: a) $\begin{cases} u(x) > 1 \\ f(x) > \varphi(x) \end{cases}$ and b) $\begin{cases} 0 < u(x) < 1 \\ f(x) < \varphi(x) \end{cases}$.

Solution of the exponential inequality is the union of solutions sets.

79. What is a logarithmic inequality?

Answer. An inequality with variables in the logarithm or in the base of the logarithm is called logarithmic inequality.

80. Which techniques for solving exponential inequalities do you know?

Answer. To solve the logarithmic inequalities it is useful to transform them to the following form: $\log_a f(x) > \log_a \varphi(x)$. This is a simple logarithmic inequality.

In solving logarithmic inequalities, it is necessary to remember the following:

- 1) expression with a variable in the logarithm can be only positive;
- 2) logarithmic function increases monotonically, if $a > 1$ and decreases monotonically, if $0 < a < 1$.

For solving some logarithmic inequalities, we have to solve a system of these inequalities.

If the base of the logarithm contains the variable $\log_{u(x)} f(x) > \log_{u(x)} \varphi(x)$, then we have to solve two systems of inequalities:

$$\text{a) } \begin{cases} u(x) > 1 \\ f(x) > 0 \\ \varphi(x) > 0 \\ f(x) > \varphi(x) \end{cases} \quad \text{b) } \begin{cases} 0 < u(x) < 1 \\ f(x) > 0 \\ \varphi(x) > 0 \\ f(x) < \varphi(x) \end{cases}$$

Solution of original inequality is a union of solutions sets.

Topic 4. TRIGONOMETRY

81. What is a straight angle?

Answer. Straight angle is the angle which sides form a straight line.

82. What is a full angle?

Answer. Full (rotation) angle or 360-degree angle is the angle that does a full circle.

83. What is a full subtended angle?

Answer. Subtended angle is the angle formed by an object at a given external point.

84. What is a full central angle?

Answer. Central angle is the angle subtended at the center of a circle by two given points on the circle. The central angle is measured by length of the arc relative to the circumference.

85. How are angles measured?

Answer. The angles are measured in degrees and radians.

86. What is one degree?

Answer. One degree is the angle subtended at the center of a circle by an arc that is equal in length to the $\frac{1}{360}$ length of the circumference.

$\frac{1}{60}$ part of degree is called a minute ($1'$).

$\frac{1}{60}$ part of the minute is called a second ($1''$).

87. What is one radian?

Answer. *One radian* is the angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle.

88. How could we transform radian to degree and vice versa?

Answer.

$$1 \text{ radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} \approx 57^\circ 17' 45'' \text{ (57 degrees, 17 minutes, 45 seconds).}$$

$$1^\circ = \frac{2\pi}{360^\circ} = \frac{\pi}{180^\circ} \text{ radians} \approx 0.017453 \text{ radians.}$$

89. What is a conversion formula from degrees to radians?

Answer. $\alpha_{\text{rad}} = \frac{\pi}{180} \cdot \alpha^\circ$.

90. What is a sine of the angle?

Answer. Sine of the angle is the ratio of the ordinate of point M_α to

the radius of the circle: $\sin \alpha = \frac{y}{R}$.

91. What is a cosine of the angle?

Answer. Cosine of the angle is the ratio of the abscissa of point M_α

to the radius of the circle: $\cos \alpha = \frac{x}{R}$.

92. What is a tangent of the angle?

Answer. Tangent of the angle α is the ratio of the ordinate of point

M_α to its abscissa: $\tan \alpha = \frac{y}{x} = \frac{\sin \alpha}{\cos \alpha}$.

The line $x = 1$ is called the axis of tangents (Fig. 36).

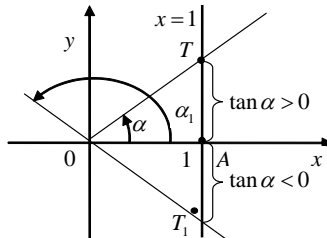


Figure 36

93. What is a cotangent of the angle?

Answer. Cotangent of the angle α is called the ratio of the abscissa

of point M_α to its ordinate: $\cot \alpha = \frac{x}{y} = \frac{\cos \alpha}{\sin \alpha}$.

The line is called the axis of cotangents (Fig. 37).

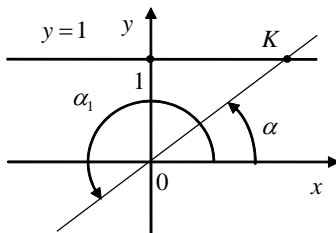


Figure 37

94. What is a secant of the angle?

Answer. Secant of the angle α has the value inversed to $\cos \alpha$:

$$\sec \alpha = \frac{1}{\cos \alpha}, \quad \cos \alpha \neq 0.$$

95. What is a cosecant of the angle?

Answer. Cosecant of the angle α has the value inversed to $\sin \alpha$:

$$\csc \alpha = \frac{1}{\sin \alpha}, \quad \sin \alpha \neq 0.$$

96. What is a basic trigonometric identity?

Answer. $\cos^2 \alpha + \sin^2 \alpha = 1$ is the basic trigonometric identity.

97. What other types of trigonometric identity do you know?

Answer. It is known other types of trigonometric identities:

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}, \quad \cot \alpha = \frac{\cos \alpha}{\sin \alpha}, \quad \cot \alpha = \frac{1}{\tan \alpha} \Rightarrow \tan \alpha \cdot \cot \alpha = 1,$$

$$1 + \tan^2 \alpha = \frac{1}{\cos^2 \alpha} \Rightarrow 1 + \tan^2 \alpha = \sec^2 \alpha,$$

$$1 + \cot^2 \alpha = \frac{1}{\sin^2 \alpha} \Rightarrow 1 + \cot^2 \alpha = \csc^2 \alpha.$$

98. What reduction formulas (the cofunction identities) do you know?

Answer. Reduction formulas are called parities which help us to transform the trigonometric functions of arguments $\frac{\pi}{2} \pm \alpha$, $\frac{3\pi}{2} \pm \alpha$, $\pi \pm \alpha$, $2\pi \pm \alpha$ to the trigonometric functions $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, $\cot \alpha$.

99. Write Sum and Difference Formulas (Identities).

Answer. Sum and Difference Formulas (Identities).

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \sin \beta \cos \alpha;$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta;$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \cdot \tan \beta} \quad (\alpha, \beta, \alpha \pm \beta \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z});$$

$$\cot(\alpha \pm \beta) = \frac{\cot \alpha \cdot \cot \beta \pm 1}{\cot \beta \pm \cot \alpha} \quad (\alpha, \beta, \alpha \pm \beta \neq n\pi, n \in \mathbb{Z}).$$

100. Write Double and Triple Angel Formulas.

Answer. Double and Triple Angel Formulas:

$$\sin 2\alpha = 2\sin \alpha \cos \alpha;$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2\cos^2 \alpha - 1 = 1 - 2\sin^2 \alpha;$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \quad (\alpha \neq \frac{\pi}{4} + \frac{\pi}{2}n, n \in \mathbb{Z}; \quad \alpha \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z});$$

$$\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha} \quad (\alpha \neq \frac{\pi}{2}n, n \in \mathbb{Z}; \quad \alpha \neq k\pi, k \in \mathbb{Z});$$

$$\sin 3\alpha = 3\sin \alpha - 4\sin^3 \alpha;$$

$$\cos 3\alpha = 4\cos^3 \alpha - 3\cos \alpha;$$

$$\tan 3\alpha = \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha} \quad (\alpha \neq \frac{\pi}{6}(2n+1), k \in \mathbb{Z});$$

$$\cot 3\alpha = \frac{3 \cot \alpha - \cot^3 \alpha}{1 - 3 \cot^2 \alpha} \quad (\alpha \neq \frac{\pi}{3}n, n \in \mathbb{Z}).$$

101. Write Power-Reducing/ Half Angle Formulas.**Answer.** Power-Reducing/ Half Angle Formulas:

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}; \quad \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}.$$

102. Write Product-to-Sum Formulas.**Answer.** Product-to-Sum Formulas:

$$\sin \alpha \cdot \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta));$$

$$\cos \alpha \cdot \cos \beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta));$$

$$\sin \alpha \cdot \cos \beta = \frac{1}{2}(\sin(\alpha - \beta) + \sin(\alpha + \beta)).$$

103. Write Sum-to-Product Formulas.**Answer.** Sum-to-Product Formulas:

$$\sin \alpha \pm \sin \beta = 2 \sin \frac{\alpha \pm \beta}{2} \cos \frac{\alpha \mp \beta}{2};$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2};$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2};$$

$$\tan \alpha \pm \tan \beta = \frac{\sin(\alpha \pm \beta)}{\cos \alpha \cos \beta} \quad (\alpha, \beta \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z});$$

$$\cot \alpha \pm \cot \beta = \frac{\sin(\alpha \pm \beta)}{\sin \alpha \sin \beta} \quad (\alpha, \beta \neq n\pi, n \in \mathbb{Z}).$$

104. Write Universal Substitution Formulas.**Answer.** Universal Substitution Formulas:

$$\sin \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} \quad (\alpha \neq \pi + n\pi, n \in \mathbb{Z});$$

$$\cos \alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} \quad (\alpha \neq \pi + n\pi, n \in \mathbb{Z});$$

$$\tan \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}} \quad (\alpha \neq \pi + 2n\pi, n \in \mathbb{Z}, \quad \alpha \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}).$$

105. Write Half-Angle Formulas.

Answer. Half-Angle Formulas:

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}; \quad \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}};$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \quad (\alpha \neq \pi + 2n\pi, n \in \mathbb{Z});$$

$$\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha} \quad (\alpha \neq n\pi, n \in \mathbb{Z});$$

$$\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha} \quad (\alpha \neq \pi + 2n\pi, n \in \mathbb{Z});$$

$$\cot \frac{\alpha}{2} = \frac{\sin \alpha}{1 - \cos \alpha} \quad (\alpha \neq 2n\pi, n \in \mathbb{Z});$$

$$\cot \frac{\alpha}{2} = \frac{1 + \cos \alpha}{\sin \alpha} \quad (\alpha \neq n\pi, n \in \mathbb{Z});$$

Topic 5. SOLVING TRIGONOMETRICAL EQUATIONS

106. What is a trigonometric equation?

Answer. Trigonometric equation is any equation that contains trigonometric function.

107. What trigonometric equations are called as the simplest trigonometric equations?

Answer. $\sin x = a$, $\cos x = a$, $\tan x = a$, $\cot x = a$ are called the simplest trigonometric equations.

108. What is a common formula for solving equation $\sin x = a$?

Answer. The common formula that allows finding all roots of the equation $\sin x = a$ is following:

$$x = (-1)^k \arcsin a + k\pi, k \in \mathbb{Z}, -\frac{\pi}{2} \leq \arcsin a \leq \frac{\pi}{2}.$$

109. What is a common formula for solving equation $\cos x = a$?

Answer. The common formula that allows finding all roots of the equation $\cos x = a$ is following:

$$x = \pm \arccos a + 2k\pi, k \in \mathbb{Z}, \text{ where } 0 \leq \arccos a \leq \pi.$$

110. What is a common formula for solving equation $\tan x = a$ and $\cot x = a$?

Answer.

$$\tan x = a \Leftrightarrow x = \arctan a + k\pi, k \in \mathbb{Z}, \text{ where } -\frac{\pi}{2} < \arctan a < \frac{\pi}{2};$$

$$\cot x = a \Leftrightarrow x = \operatorname{arccot} a + k\pi, k \in \mathbb{Z}, \text{ where } -\frac{\pi}{2} < \operatorname{arccot} a < \frac{\pi}{2}.$$

111. What stages does solving of trigonometric equations consist of?

Answer. Solving of trigonometric equations consists of the two stages: transforming an equation to receive its simplest shape and solving the received simplest trigonometric equation. There are seven basic methods of solving trigonometric equations: algebraic method, reducing to a homogeneous equation, factoring, power reducing method, transforming a product to a sum, introducing an auxiliary angle, universal substitution, substitution of the form $\sin x \pm \cos x = t$.

112. What do you know about algebraic method?

Answer. Algebraic method is well known from algebra (substitution method). In solving equations it is commonly used the basic trigonometric identity: $\sin^2 x + \cos^2 x = 1$.

113. What do you know about method of reducing to a homogeneous equation?

Answer. An equation is called a *homogeneous equation* with respect to $\sin x$ and $\cos x$, if all its terms are of the same degree.

To solve the homogeneous equation, it is necessary:

- a) to transfer all terms to the left-hand side;
- b) to take all common factors out of parentheses;
- c) to equate all factors and parentheses to zero;
- d) the parentheses equated to zero give the homogeneous equation of smaller degree, which should be divided by cosine (or sine) of the higher power;
- e) to solve the received algebraic equation with respect to tangent x .

114. What do you know about factoring method?

Answer. Factoring is based on the fact that the equation

$f(x) \cdot \varphi(x) = 0$ is equivalent to the set of equations $\begin{cases} f(x) = 0 \\ \varphi(x) = 0 \end{cases}$ from

the domain of the $f(x) \cdot \varphi(x) = 0$ equation.

115. What do you know about power reducing method?

Answer. In solving the equations it is used the formulas of the power

reducing: $\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$, $\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$.

116. What do you know about transforming a product to a sum?

Answer. Transforming a Product to a Sum Formulas are used in this case:

$$\sin \alpha \cos \alpha = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2};$$

$$\sin \alpha \sin \beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2};$$

$$\cos \alpha \cos \beta = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}.$$

117. What do you know about Universal Substitution?

Answer. According to the method of universal substitution, it is helpful to use following formulas:

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \quad \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \quad \tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}, \quad \cot x = \frac{1 - \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}}.$$

118. What do you know about Substitution of the form $\sin x \pm \cos x = t$?

Answer. Substitution of the form $\sin x \pm \cos x = t$. In this case, we have two substitutions:

$$1) \begin{cases} \sin x + \cos x = t \\ \sin x \cdot \cos x = \frac{t^2 - 1}{2} \end{cases} \quad 2) \begin{cases} \sin x - \cos x = t \\ \sin x \cdot \cos x = \frac{1 - t^2}{2} \end{cases}.$$

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