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NATIONAL TECHNICAL UNIVERSITY

"KHARKIV POLYTECHNIC INSTITUTE"

METHODICAL RECOMMENDATIONS FOR WORKSHOPS

Topic: «Resource Allocation Optimization Problem»

academic discipline: «Computer Mathematics. Part 1»

for students of specialty 121 "Software Engineering"

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Introduction

Methodical recommendations and tasks for self-study activity on the topic "Resource Allocation Optimization Problem" are presented. A list of recommended literature is provided.

Methodical guidelines

All problems within given theme are solved using Solver Add-in. The Solver Add-in is a Microsoft Office Excel add-in program that is available when you install Microsoft Office or Excel. To use the Solver Add-in, however, you first need to load it:

1. Click the **Microsoft Office Button**, and then click **Excel Options**.
2. Under **Add-ins**, select **Solver Add-in** and click on the **Go** button (Fig. 1).
3. In the Add-Ins available box, click Solver Add-in and click OK (Fig. 2).

If **Solver Add-in** is not listed in the **Add-Ins available** box, click **Browse** to locate the add-in. If you get prompted that the Solver Add-in is not currently installed on your computer, click **Yes** to install it.

4. After you load the Solver Add-in, the **Solver** command is available in the **Analysis** group on the **Data** tab (Fig. 3).

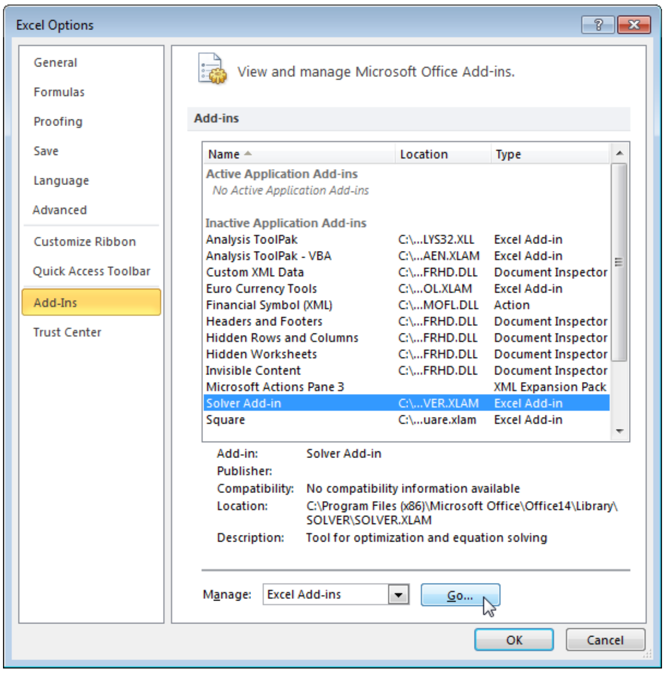


Fig.1. Excel Options

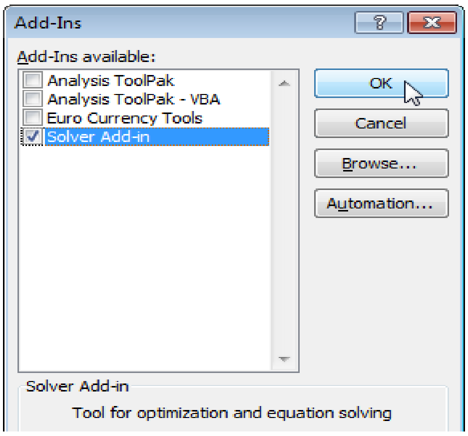


Fig.2. Add-Ins

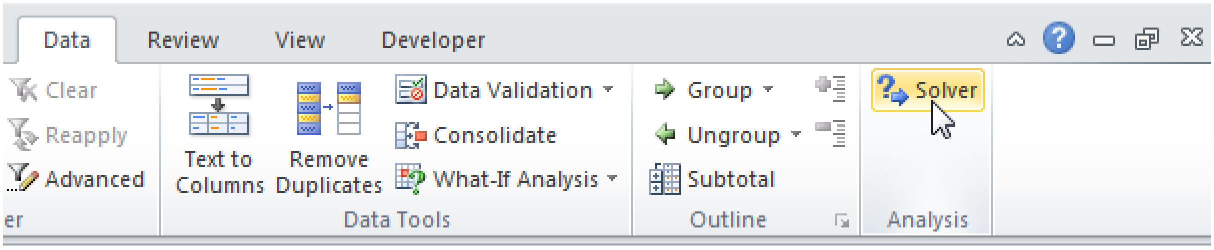


Fig.3. Solver

Resource Allocation Optimization Problem

The linear programming (LP) model has three basic components.

1. Decision variables that we seek to determine.

2. Objective (goal) that we need to optimize (maximize or minimize).

3. Constraints that the solution must satisfy.

In order to solve a linear programming problem using the simplex method, it must be presented in canonical form:

$$F = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \to \max$$

$$\begin{cases} x_1 + a_{1,m+1} x_{m+1} + \ldots + a_{1n} x_n = b_1 \\ x_2 + a_{2,m+1} x_{m+1} + \ldots + a_{2n} x_n = b_2 \\ \vdots \\ x_m + a_{m,m+1} x_{m+1} + \ldots + a_{mn} x_n = b_m \end{cases}$$

$$x_1, x_2, \ldots, x_n \geq 0$$

If the task is written in canonical form, you can always specify a current basic solution for it: $X = (x_1 = b_1, x_2 = b_2, \ldots, x_m = b_m, x_{m+1} = x_{m+2} = \ldots = x_n = 0)$.

Next, applying the simplex method algorithm, the current basic solution is checked for optimality and the optimal decision is found.

The simplex method algorithm includes the following steps:

1. Fill in lines from 1 to m of the simplex table (see Table 1):

Table 1

Simplex table

i	Basis	C_b	P_0	c_1	c_2	…	c_r	…	c_m	c_{m+1}	…	c_k	…	c_n
				P_1	P_2	…	P_r	…	P_m	P_{m+1}	…	P_k	…	P_n
1	P_1	c_1	b_1	1	0	…	0	…	0	a_{1m+1}	…	a_{1k}	…	a_{1n}
2	P_2	c_2	b_2	0	1	…	0	…	0	a_{2m+1}	…	a_{2k}	…	a_{2n}
…	…	…	…	…	…	…	…	…	…	...	…	…	…	…
r	P_r	c_r	b_r	0	0	…	1	…	0	a_{rm+1}	…	a_{rk}	…	a_{rn}
…	…	…	…	…	…	…	…	…	…	...	…	…	…	…
m	P_m	c_m	b_m	0	0	…	0	…	1	a_{mm+1}	…	a_{mk}	…	a_{mn}
m+1			F_0	0	0	…	0	…	0	Δ_{m+1}	…	Δ_k	…	Δ_n

2. Calculate cell values of the (m+1)-th row:

$$\Delta_j = \sum_{i=1}^{m} c_i a_{ij} - c_j, \quad F_0 = \sum_{i=1}^{m} c_i b_i,$$

F_0 - current objective function value.

The optimality criterion is the non-negativity of all values of Δ_j.

If $\Delta_j < 0$ for some j and all values corresponding to this index $a_{ij} \le 0$, then the problem has no solution. Otherwise, the simplex table values are to be recalculated.

3. Find pivot column and pivot row. The pivot column k is determined by the largest absolute value among negative values of Δ_j. X_k is known as the entering variable. The mechanics for determining the leaving variable X_r calls for computing the ratios of the right-hand side of the equations to the corresponding (strictly) positive

constraint coefficients under the entering variable X_k. The r-th row is called pivot row. As a result, the pivot element is situated on the intersection of pivot row and pivot column.

4. The elements of the new simplex table are determined by the formulas:

$$b'_i = \begin{cases} b_i - \dfrac{b_r a_{ik}}{a_{rk}}, i \neq r \\ \dfrac{b_r}{a_{rk}}, i = r \end{cases} \qquad a'_{ij} = \begin{cases} a_{ij} - \dfrac{a_{rj} a_{ik}}{a_{rk}}, i \neq r \\ \dfrac{a_{rj}}{a_{rk}}, i = r \end{cases}$$

5. Then the found solution is checked again for optimality.

Example 1. Reddy Mikks produces both pink and yellow paints from two raw materials, M1 and M2. The following table provides the basic data of the problem:

Table 2

Initial data

	Tons of raw material per ton of		
	Yellow paint	Pink paint	Maximum daily availability (tons)
Raw material, M1	6	4	24
Raw material M2	1	2	6
Profit per ton ($1000)	5	4	

Reddy Mikks wants to determine the optimum (best) product mix of Yellow and Pink paints that maximizes the total daily profit.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of each paint. Thus the variables of the model are defined as:

x_1 - tons produced daily of yellow paint, x_2 - tons produced daily of pink paint.

To construct the objective function, note that the company wants to maximize the total daily profit of both paints. Given that the profits per ton of yellow and pink paints are 5 and 4 (thousand) dollars, respectively, it follows that letting F represent the total daily profit, the objective of the company is

$$F = 5x_1 + 4x_2.$$

Next, we construct the constraints that restrict raw material usage and product demand. Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$6x_1 + 4x_2 \leqslant 24$ (Raw material MI)

$1x_1 + 2x_2 \leqslant 6$ (Raw material M2)

A hidden (or "understood-to-be") restriction is that variables x_1 and x_2 cannot assume negative values:

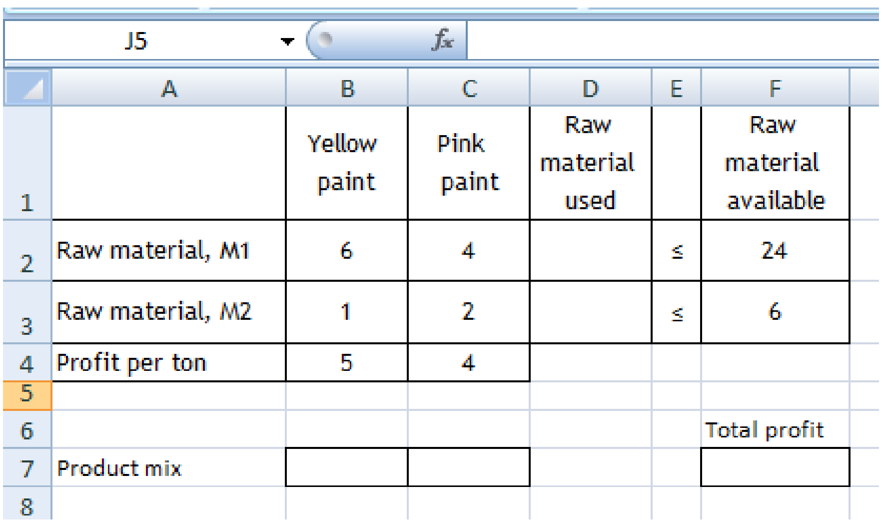
$x_1 \geq 0$, $x_2 \geq 0$

The complete Reddy Mikks model is

$F = 5x_1 + 4x_2 \rightarrow \max$

$$\begin{cases} 6x_1 + 4x_2 \leq 24 \\ 1x_1 + 2x_2 \leq 6 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

The model we are going to solve looks as follows in Excel (Fig. 4).



	A	B	C	D	E	F
1		Yellow paint	Pink paint	Raw material used		Raw material available
2	Raw material, M1	6	4		≤	24
3	Raw material, M2	1	2		≤	6
4	Profit per ton	5	4			
5						
6						Total profit
7	Product mix					
8						

Fig.4. Initial data

Insert the following three SUMPRODUCT functions (Fig. 5).

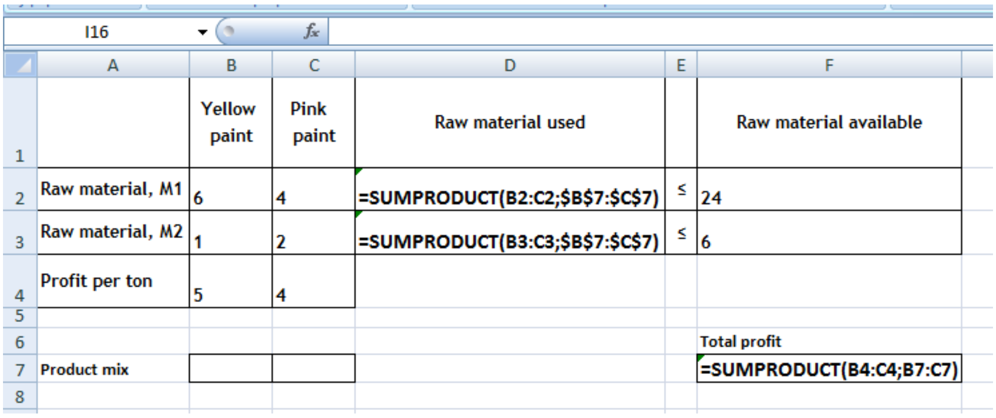


Fig.5. Functions to be inserted

To find the optimal solution, execute the following steps.

1. On the Data tab, click Solver (see Fig.6).

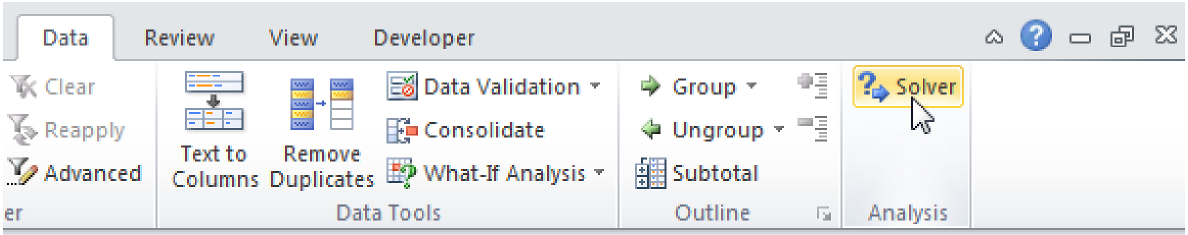


Fig.6.

Enter the solver parameters. The result should be consistent with the Fig. 7.

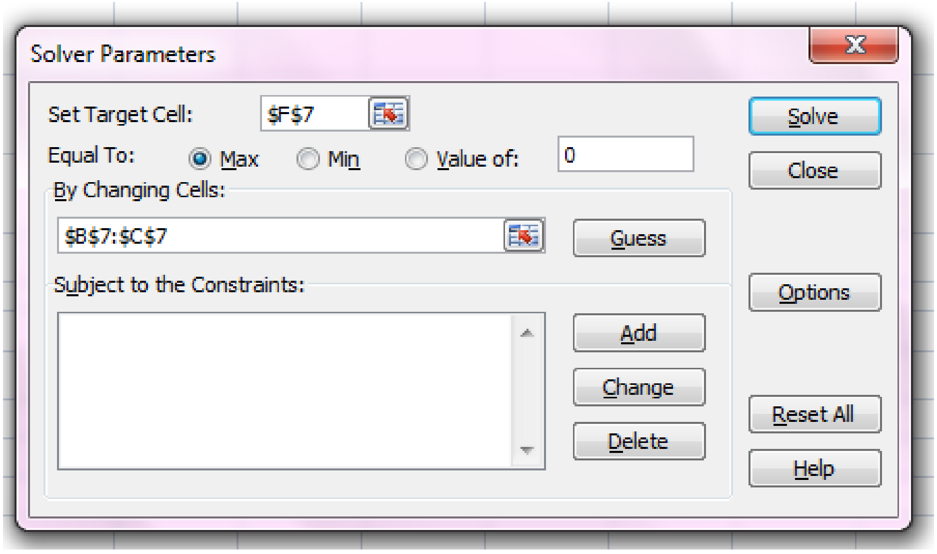


Fig.7. Solver Parameters

2. Enter F7 for the Objective.

3. Click Max.

4. Enter B7:C7 for the Changing Variable Cells.

5. Click Add to enter the constraints (see Fig.8).

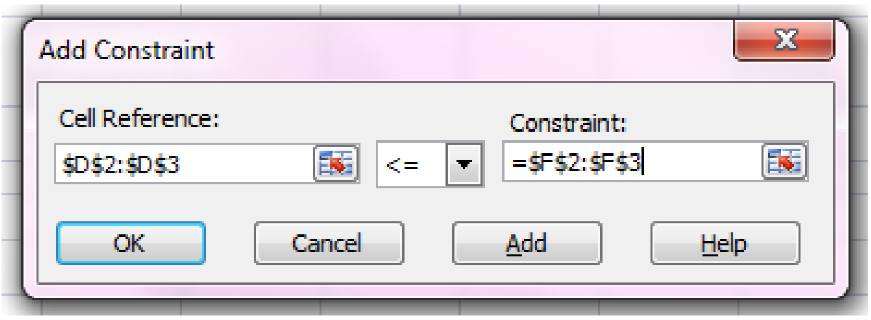


Fig.8. Add Constraint

6. Click Options and select “Assume Linear Model” and “Assume Non-Negative” (Fig. 9).

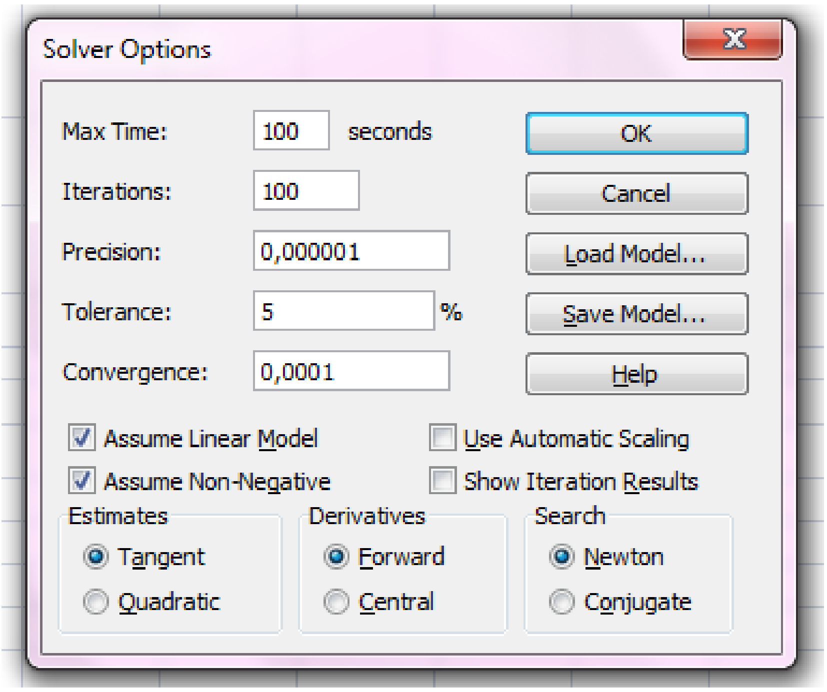


Fig. 9. Solver Options

7. Finally, click Solve. Result is shown on the Fig. 10. Choose “Answer” and “Sensitivity”.

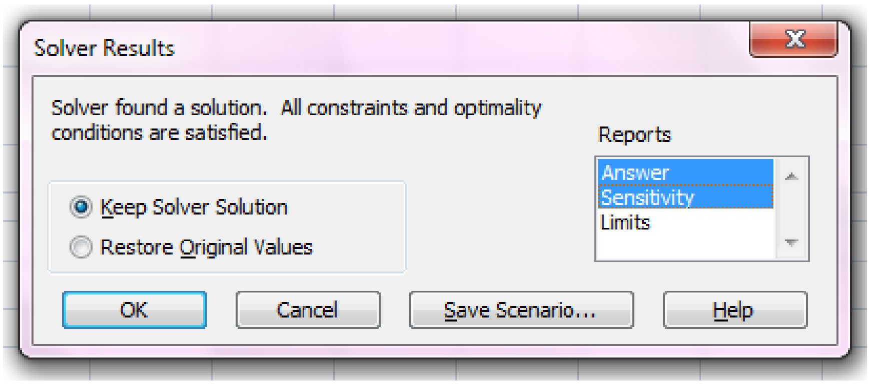


Fig. 10. Solver Results

The optimal solution is shown below (see Fig.11).

Conclusion: it is optimal to produce 3 tons of yellow paint and 1,5 tons of pink paint. This solution gives the maximum profit of 21. This solution uses all the resources available.

	A	B	C	D	E	F
1		Yellow paint	Pink paint	Raw material used		Raw material available
2	Raw material, M1	6	4	24	≤	24
3	Raw material, M2	1	2	6	≤	6
4	Profit per ton	5	4			
5						
6						Total profit
7	**Product mix**	3	1,5			21

Fig.11. Optimal solution

The same information may be obtained from the Answer report (see Fig. 12).

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$F$7	Profit per ton ($1000)	0	21

Variable Cells

Cell	Name	Original Value	Final Value
$B$7	Yellow paint	0	3
$C$7	Pink paint	0	1,5

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$4	Raw material, M1	24	$E$4<=$D$4	binding	0
$E$5	Raw material, M2	6	$E$5<=$D$5	binding	0

Fig.12. Answer report

From the "Constraints" table you can determine whether raw materials are used fully or not. If raw material has "binding" status in appropriate column, it is used fully for optimal product mix.

Sensitivity analysis gives insight in how the optimal solution changes when you change the coefficients of the model (Fig. 13).

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$7	Yellow paint	3	0	5	1	3
$C$7	Pink paint	1,5	0	4	6	0,666666667

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$2	Raw material, M1	24	0,75	24	12	12
$D$3	Raw material, M2	6	0,5	6	6	2

Fig.13. Sensitivity report

Let's discuss "Variable Cells" table.

If Reduced Cost equals 0, the appropriate type of product has nonzero value. If Reduced Cost doesn't equal 0, it is not beneficial to include the appropriate type of product into the optimal product mix.

Suppose that the unit profit of yellow paint is fixed at its current value of $5. The optimality range for unit profit of pink paint, that keeps the optimum solution unchanged, may be obtained from the last two columns: [4-0,66; 4+6].

Suppose that the unit profit of pink paint is fixed at its current value of $4. The optimality range for unit profit of yellow paint, that keeps the optimum solution unchanged, may be obtained from the last two columns: [5-3; 5+1].

Let's discuss "Constraints" table.

The shadow prices tell us how much the optimal solution can be increased or decreased if we change the right hand side values (resources available) with one unit. This means that a unit increase in the availability of raw material M1 will increase profit by $0.75; a unit increase in the availability of raw material M2 will increase profit by $0.5.

The dual price of $0.75 remains valid for changes in M1 availability within the following boundaries [24-12; 24+12]. The dual price of $0.5 remains valid for changes in M2 availability within the following boundaries [6-2; 6+6]. You can take these values from the last two columns.

Final value column in "constraints" table shows the used amounts of each raw material. You can see that both resources are used fully.

A company produces two products. Both products use four raw materials. The maximum daily availability for each material and raw materials usage rates per unit of product 1 and per unit of product 2 are represented in the table. Determine the optimal product mix that maximizes the total daily profit.

Example 2. Let’s solve the task from Example 1 with additional limitations:

- The daily demand for pink paint cannot exceed that for yellow paint by more than 1 ton.
- The maximum daily demand for pink paint is 2 tons.

Then the model can be rewritten as follows:

$$F = 5x_1 + 4x_2 \rightarrow \max$$

$$\begin{cases} 6x_1 + 4x_2 \le 24 \\ 1x_1 + 2x_2 \le 6 \\ x_2 \le 2 \\ x_2 - x_1 \le 1 \\ x_1 \ge 0 \\ x_2 \ge 0 \end{cases}$$

Three consequent simplex tables are presented in Table 3– Table 5.

Table 3

The first simplex table

				5	**4**	**0**	**0**	**0**	**0**	
	Basis	**Cb**	**P0**	**P1**	**P2**	**P3**	**P4**	**P5**	**P6**	
1	P3	0	24	**6**	4	1	0	0	0	24 / 6=4
2	P4	0	6	1	2	0	1	0	0	6 / 1=6
3	P5	0	1	-1	1	0	0	1	0	
4	P6	0	2	0	1	0	0	0	1	
5			0	-5	-4	0	0	0	0	

Current basic solution is $X = (x_1 = 0, x_2 = 0, x_3 = 24, x_4 = 6, x_5 = 1, x_6 = 2)$.

Analyzing the data of the fifth row, we see that the solution is not optimal and are to be recalculated (see Table 4).

Table 4

The second simplex table

	Basis	Cb	P0	5	4	0	0	0	0	
				P1	**P2**	**P3**	**P4**	**P5**	**P6**	
1	P1	5	4	1	2/3	1/6	0	0	0	6
2	P4	0	2	0	**4/3**	-1/6	1	0	0	3/2
3	P5	0	5	0	5/3	1/6	0	1	0	3
4	P6	0	2	0	1	0	0	0	1	2
5			20	0	-2/3	5/6	0	0	0	

Current basic solution is $X = (x_1 = 4, x_2 = 0, x_3 = 0, x_4 = 2, x_5 = 5, x_6 = 2)$

Analyzing the data of the fifth row, we see that the solution is not optimal and are to be recalculated (see Table 5).

Table 5

The third simplex table

	Basis	Cb	P0	5	4	0	0	0	0	
				P1	**P2**	**P3**	**P4**	**P5**	**P6**	
1	P1	5	3	1	0	1/4	-1/2	0	0	
2	P2	4	3/2	0	1	-1/8	¾	0	0	
3	P5	0	5/2	0	0	3/8	-5/4	1	0	
4	P6	0	1/2	0	0	1/8	-3/4	0	1	
5			21	0	0	¾	1/2	0	0	

Optimal solution is found: $X = (x_1 = 3, x_2 = 3/2, x_3 = 0, x_4 = 0, x_5 = 5/2, x_6 = 1/2)$, $F(X) = 21$.

Tasks for self-study work

A company produces two products. Both products use four raw materials. The maximum daily availability for each material and raw materials usage rates per unit of product 1 and per unit of product 2 are represented in the appropriate table below.

Choose your variant and determine the optimal product mix that maximizes the total daily profit.

Variant 1

Raw Material	Tons of raw material per ton of		Material Available
	Product 1	Product 2	
A	4	1	7
B	1	2	10
C	3	1	6
D	6	1	10
Profit per ton	7	2	

Variant 2

Raw Material	Tons of raw material per ton of		Material Available
	Product 1	Product 2	
A	0	1	4
B	4	1	7
C	2	1	5
D	6	1	10
Profit per ton	4	3	

Variant 3

Raw Material	Tons of raw material per ton of		Material Available
	Product 1	Product 2	
A	1	0	2
B	1	1	6
C	2	1	7
D	4	1	10
Profit per ton	3	2	

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