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Modelling and Nonlinear Dynamics of Third-Order Thermomechanically Coupled Laminated Plates

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Abstract
Thermomechanically coupled, geometrically nonlinear, laminated plates are addressed through a unified 2D formulation that integrates mechanical and thermal aspects and consistently accounts for cubic variations along the thickness of both in-plane displacement components and temperature. It allows to address a variety of thermal boundary conditions on the plate upper and lower surfaces. Minimal dimension reduction of the problem is pursued for symmetric cross-ply laminates. A numerical case study provides hints on the potential of the reduced model for the analysis of thermomechanical coupling effects on the system nonlinear dynamics.
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Keywords
Thermomechanical modelling, laminated plates, nonlinear dynamics

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Introduction

Thermoelastic analysis of composite plates and shells has been addressed in the last two decades by a variety of models, mostly via numerical approaches aimed at evaluating the effect of partial or full thermomechanical coupling on either statics and linear dynamics or actual nonlinear dynamics.

Yet, in view of developing a comprehensive modelling framework suitable to accomplish extended, yet computationally handable and controllable, investigations of the fundamental aspects of thermomechanically coupled nonlinear dynamic behavior of 2D composite structures, two main aspects of particular interest seem to be somehow overlooked in the literature.

(i) The need of clearly integrating mechanical and thermal aspects within a unified continuum formulation of the 2D coupled problem, thus allowing for an aware evaluation of possibly needed, yet consistent, simplifying assumptions, without merely accompanying the classical (purely mechanical) 2D plate structural theories [1, 2] with occasional and non-systematic treatments [3] of the thermoelastic coupling, or simply adapting the 3D relationships of heat transfer and thermoelasticity to the problem at hand.

(ii) The suitability of developing reduced order models (ROMs) which still preserve the main nonlinear dynamic features of the underlying continuum formulation but also allow for an easier analysis and understanding of the basic, yet possibly involved, effects of full or partial coupling on the finite amplitude vibrations of geometrically nonlinear laminated plates. Thus getting rid, at least in the initial research stage, of the complicatedness often ensuing from the use of much richer models (e.g., finite elements, also possibly implemented within an effective unified perspective [4]) in the interpretation of nonlinear phenomena.

The first goal can be achieved by exploiting the classical Tonti diagram [5] for physical theories to identify generalized 2D variables and governing equations for both mechanical and thermal aspects of the problem, working in parallel. In fact, Tonti's approach allows us to identify and compare a multitude of possible continuous models of thermomechanical laminated plates resulting from different aware assumptions about the involved configurations and quantities [6], as well as to implement them for two-field analyses of system nonlinear dynamics. This approach has been effectively used in [7] to implement a unified 2D formulation of the Classical von Karman theory with Thermomechanical Coupling (CTC), with assumed linear temperature variation along the laminate thickness.

As regards the second goal, just few works have used low-order models to analyse the basic effects of actual thermomechanical coupling on plate nonlinear vibrations, referring to either isotropic [8] or orthotropic [3] plates.

For symmetric cross-ply laminates, by referring to the mentioned CTC formulation, a three-mode minimal model described by nonlinear ordinary differential equations (ODEs) with one mechanical and two thermal time-dependent variables has been obtained in [7] via the Galerkin procedure. The ROM has been used to describe and compare nonlinear vibrations of the laminate under various modelling assumptions/simplifications, as well as physical conditions, and to evaluate the importance of different equation terms inducing thermal effects.

However, shear deformability is known to also play a meaningful role in the nonlinear response of laminated plates, as highlighted by the analysis of both multimode discretized models [9] and ROMs [10]. To account for its effect within a more general thermomechanical context, the present work makes a step ahead in the direction of a combined unified and comprehensive formulation of the continuum problem and minimal dimension reduction of the discretized problem. A Third order theory of shear deformable von Karman laminated plates with Thermomechanical Coupling (TTC), which encompasses the mechanical Reddy theory [1] and the classical equations of thermal nature [11], is developed via Tonti's modelling approach. Consistent with the assumed cubic variation of the displacement field along the thickness coordinate [1], a corresponding cubic variation over the thickness is assumed for the temperature field, parallel to what previously accomplished in [7] where both physical fields were assumed to vary linearly in the thickness direction.

1. Unified Modelling of a Third Order Nonlinear Laminated Plate with Thermomechanical Coupling

The unified scheme for the formulation of the nonlinear dynamic problem of thermomechanically coupled laminated plates [7] is presented in Fig.1. It integrates mechanical and thermal aspects by identifying generalized 2D variables, phenomenological quantities and governing equations also for the thermal aspects of the problem. The formulation virtually embeds a multitude of possible models, resulting from different mechanical and thermal assumptions [6].

The unified scheme is used to construct a 2D TTC model of laminated rectangular plates subjected to distributed transverse mechanical excitation, variable thermal loadings, and stretching forces acting in the x, y mid-plane of the plate on its geometrical edges.

Consistent with the assumption of a cubic function of the thickness coordinate for the 3D displacement field [1], a cubic function is assumed also for the 3D temperature field

$$T = T_0 + zT_1 + z^2T_2 + z^3T_3 \tag{1}$$

where $T(x,y,z,t)$ is the temperature variation with respect to the reference one, and $T_0(x,y,t)$, $T_1(x,y,t)$, $T_2(x,y,t)$, $T_3(x,y,t)$ are unknown temperature components of the 2D plate model. T_2 and T_3 can be expressed in terms of T_0 and T_1 by imposing a variable (either pure or mixed) combination of the following thermal boundary conditions on the upper and lower surfaces of the plate [11]:

$T\big\|_{z=\pm h/2} = T^*(x,y,t)$	temperature prescribed	(2a)
$\left.\dfrac{\partial T}{\partial z}\right\|_{z=\pm h/2} = 0$	thermal insulation	(2b)
$q_3\big\|_{z=\pm h/2} = q_3^*(x,y,t)$	heat flow prescribed	(2c)
$q_3\big\|_{z=\pm h/2} = \pm H\left[T_\infty - (T)_{\pm h/2}\right]$	free heat exchange	(2d)

where q_3 is the heat flow in the z transverse direction, H is the boundary conductance, T_∞ is the constant difference between the absolute temperature of the surrounding medium and the reference temperature, and T^* and q_3^* are the temperature and heat flow prescribed on the external surfaces, respectively. As a result, the temperature cubic profile (1) can be expressed as:

$$T = f_a(z)T_0 + f_b(z)T_1 + f_c(z) \tag{3}$$

where

$$f_a(z) = \left(r_1 + r_2 z + r_3 z^2 + r_4 z^3\right), \quad f_b(z) = \left(r_5 + r_6 z + r_7 z^2 + r_8 z^3\right), \quad f_c(z) = \left(r_9 + r_{10} z + r_{11} z^2 + r_{12} z^3\right) \tag{4}$$

with the coefficients r_i depending on the kind of imposed thermal boundary conditions.

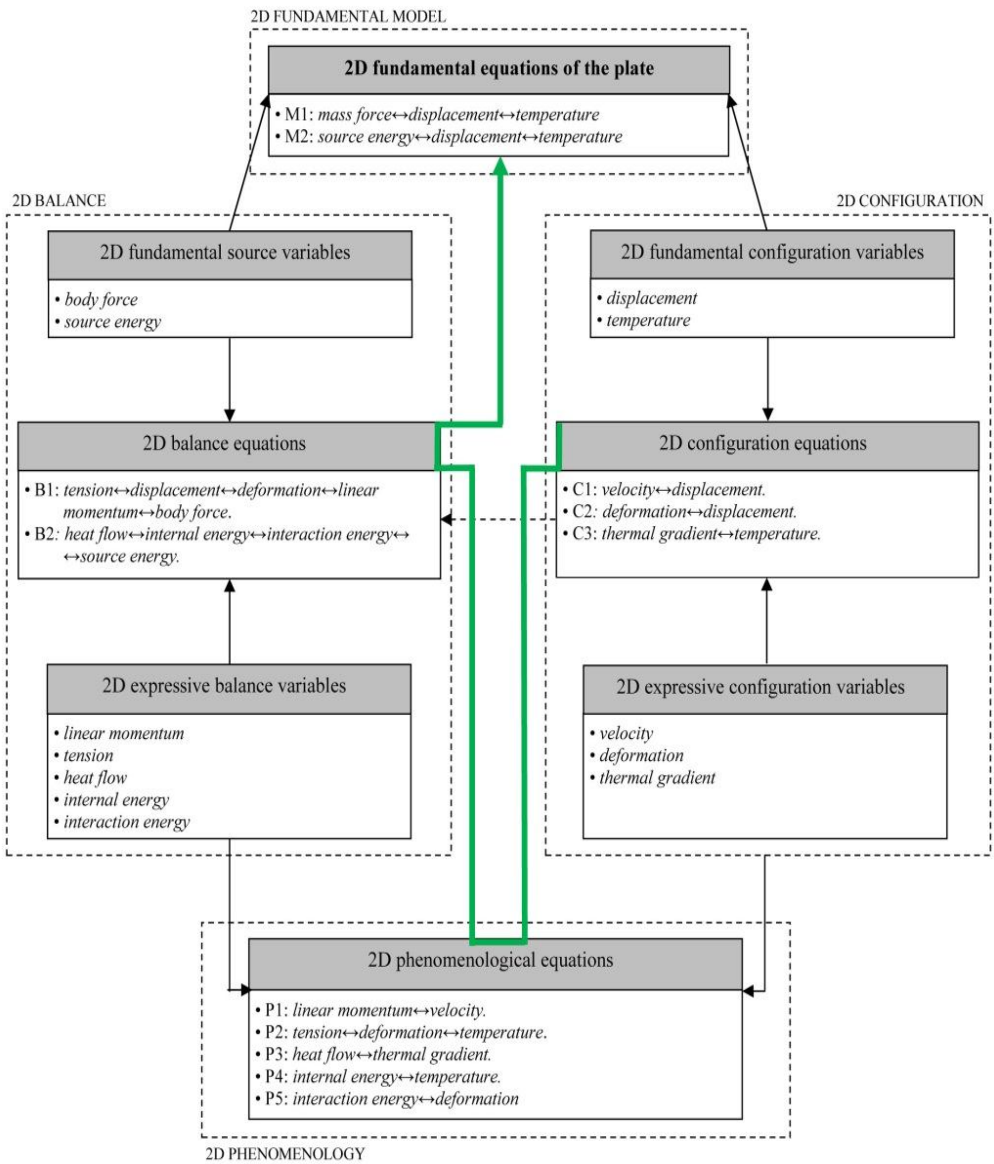


Figure 1. Unified modelling scheme for the nonlinear dynamics of the 2D thermomechanical plate

By properly combining the configuration, phenomenological, and balance relations schematically indicated in Fig. 1, the mechanical and thermal partial differential equations (PDEs) of motion of the TTC model of thermoelastic laminated plate are obtained in terms of the unknown 2D configuration variables (three mid-plane displacement components, u, v, w; two plate thickness shear-rotations, ϕ_1, ϕ_2; membrane and bending temperatures, T_0, T_1), along with the corresponding boundary conditions. Contrary to the case of a linear temperature profile assumption [7], the considered cubic temperature profile (Eqs. (3) and (4)) allows us to account for all kinds of thermal boundary conditions to be possibly prescribed on the external plate surfaces.

2. Minimal dimension reduction

In the context of a minimal Galerkin discretization to be possibly pursued in conditions of no internal resonance between the transverse modes of the laminate, single-mode approximations are assumed for the five 2D configuration variables w, ϕ_1, ϕ_2, T_0, T_1 (transverse displacement, shear-rotations, and temperatures)

$$w(x,y,t) = W(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{5a}$$

$$\phi_1(x,y,t) = \phi_{R1}(t)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{5b}$$

$$\phi_2(x,y,t) = \phi_{R2}(t)\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \tag{5c}$$

$$T_0(x,y,t) = T_{R0}(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{5d}$$

$$T_1(x,y,t) = T_{R1}(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{5e}$$

where W, ϕ_{R1}, ϕ_{R2}, T_{R0}, T_{R1} are unknown time-dependent reduced variables, whereas $\sin(\pi x/a)\sin(\pi y/b)$, $\cos(\pi x/a)\sin(\pi y/b)$ and $\sin(\pi x/a)\cos(\pi y/b)$ are shape functions satisfying both mechanical and thermal boundary conditions on the plate edges.

As regards the in-plane displacement components u, v, they are given the expressions

$$u(x,y,t) = u_{11}^{T}T_{R0}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + u_{20}^{w}W^2\sin\frac{2\pi x}{a} + u_{22}^{w}W^2\sin\frac{2\pi x}{a}\cos\frac{2\pi y}{b} + \left(u_c^{w}W^2 + u_c^{T,p}\right)\left(1-\frac{2x}{a}\right) \tag{6a}$$

$$v(x,y,t) = v_{11}^{T}T_{R0}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} + v_{02}^{w}W^2\sin\frac{2\pi y}{b} + v_{22}^{w}W^2\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b} + \left(v_c^{w}W^2 + v_c^{T,p}\right)\left(1-\frac{2y}{b}\right) \tag{6b}$$

which ensue from the kinematic condensation of the corresponding (in-plane) PDEs of motion (under the usual assumption of a frequency of in-plane vibration much higher than the frequency of transverse vibration), and also satisfy the mechanical boundary conditions. This allows us to express $u(x, y, t)$ and $v(x, y, t)$ in terms of W, T_{R0}, via known displacement coefficients, and to obtain a system of five ODEs in the unknowns W, ϕ_{R1}, ϕ_{R2}, T_{R0}, T_{R1}. Note that the expressions (6a, b) hold for symmetric cross-ply laminates, with the last terms which multiply $(1-2x/a)$ and $(1-2y/b)$ only occurring in the case of in-plane fully movable plate edges.

Then, a further kinematic condensation of the two shear-rotation equations, performed at the discretized level [10] for cross-ply laminates, allows us to link the relevant reduced variables ϕ_{R1} and ϕ_{R2} to the reduced variables W, T_{R1}, overall ending up to an ODE system with only three unknown reduced components (W, T_{R0} and T_{R1}). This is a minimal ROM still exhibiting some main coupling terms between mechanical and thermal variables, and being thus suitable for the analysis of some basic relevant effects on the nonlinear dynamics of thermomechanical laminated plates. The model has the same structure as that of the ROM based on the CTC continuous model with linear temperature profile assumption [7], if referred to a laminate in the same mechanical and thermal conditions. The differences between the two models stand in the expressions of the coefficients, which in the current TTC model are considerably more involved because of incorporating also the higher order displacement and temperature assumptions.

In the analysis, variable mechanical and thermal boundary conditions can be considered at the plate edges and on its external surfaces, respectively, giving rise to three ODEs with different terms indirectly or directly coupling the mechanical and thermal equations with each other. Edges are movable in-plane, with either free or fixed corners. Upper and lower external surfaces can be subjected to whatever thermal condition (see Eq. (2)), contrary to the CTC model [7] with which, due to the linear temperature profile assumption, only the free heat exchange condition can be analyzed.

By way of example, referring to a symmetric cross-ply laminate with simply supported and isothermal movable edges, with fixed corners, and considering a dome-shaped heat flow

$$q^{*}(x,y,t)=q(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{7}$$

prescribed on the upper surface, along with free heat exchange on the lower surface, the following ODE system is obtained:

$$\begin{aligned}
&a_{11}\ddot{W}+a_{12}\dot{W}+\left(a_{13}+a_{14}T_{R0}+a_{15}T_{R1}\right)W+a_{16}W^{3}+a_{17}T_{R1}+a_{18}T_{R0}=a_{19}F_{3}^{(0)}\\
&a_{21}\dot{T}_{R0}+a_{22}T_{R0}+a_{23}T_{R1}+a_{24}\dot{W}\cdot W+a_{25}T_{\infty}+a_{26}q(t)+a_{27}\dot{q}(t)=a_{28}E^{(0)}\\
&a_{31}\dot{T}_{R1}+a_{32}T_{R1}+a_{33}T_{R0}+a_{34}T_{\infty}+a_{35}\dot{W}+a_{36}q(t)+a_{37}\dot{q}(t)=a_{38}E^{(1)}
\end{aligned} \tag{8}$$

where the a_{ij} coefficients depend on the mechanical and thermal properties of the laminated plate, $F_3^{(0)}$ is the resulting mechanical transverse force, and $E^{(0)}$, $E^{(1)}$ are the membrane and bending thermal excitations, respectively.

3. A case study of nonlinear thermomechanically coupled oscillations

We refer to a rectangular symmetric cross-ply laminate of thickness $h=0.01\,\text{m}$ and edge lengths $a=b=1\,\text{m}$, composed of eight orthotropic unidirectional fiber-reinforced laminae of equal thickness and properties, subjected to the following mechanical (transverse) harmonic excitation

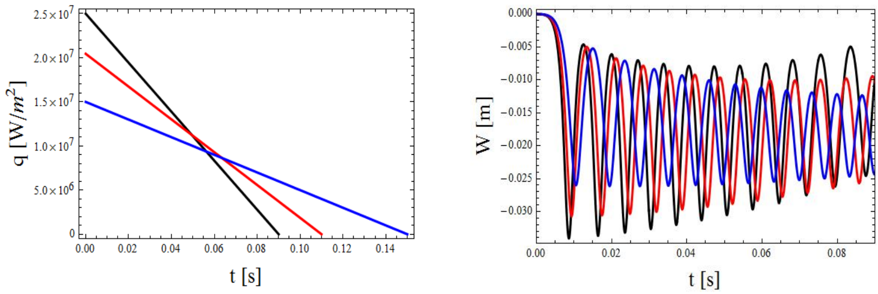
$$F_3^{(0)}=f\cos\Omega t \tag{9}$$

acting at primary resonance with the plate linear frequency.

The case of a heat impulse on the upper plate surface and free heat exchange on the lower surface, for which Eq. (8) holds, is considered. Similar analyses via a different procedure were performed in the case of Timoshenko isotropic beam [12]. Figure 2a presents three impulses of different duration t_0 that are supposed to be spatially distributed along the plate according to the dome-shape of Eq. (7), with an amplitude decreasing in time as:

$$q(t)=\overline{q}\left(1-t/t_0\right) \tag{10}$$

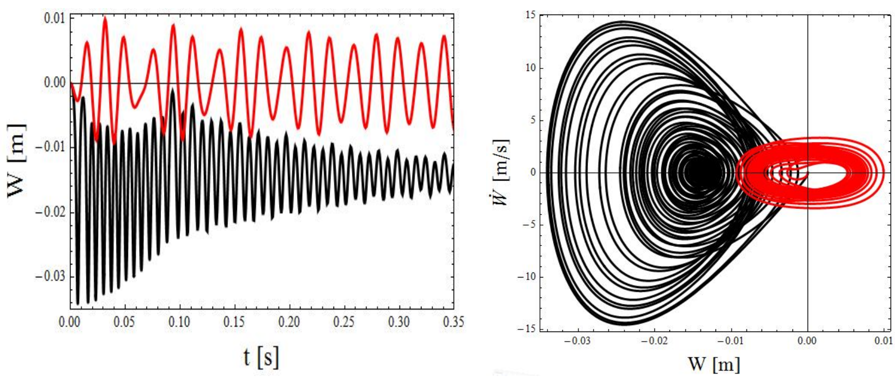
and $\overline{q}\times t_0=\text{constant}$. Figure 2b shows their influence on the initial plate deflection. As expected, shorter impulses cause vibrations with larger amplitudes, because, in a fixed short time, more energy is transmitted to the plate.



(a) (b)

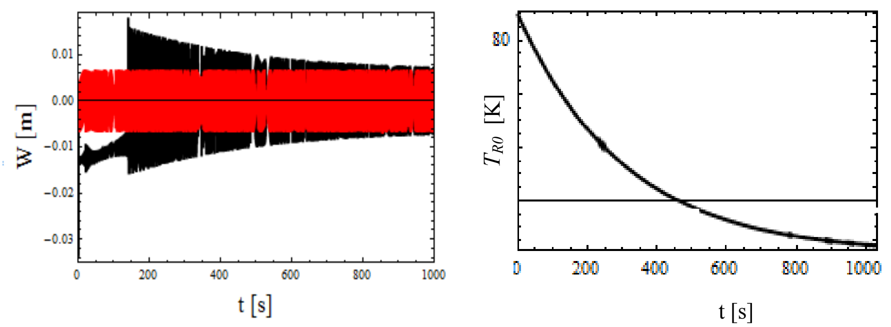
Figure 2. Time histories of the impulses (a) and of the ensuing plate deflection (b)

Figure 3 compares the time history and the phase portrait of the deflection W when the heat impulse (10) is acting together with a mechanical forcing (9) (black lines) versus the case when there is only mechanical forcing (red lines). The heat loading significantly changes the character of the plate response: an additional plate bending caused by non-uniform temperature distribution along the thickness induces vibrations around a new, shifted, equilibrium and larger maximal deflections. Indeed, the critical buckling temperature has been largely overcome, and the characteristics of the transverse mechanical excitation do not allow the plate vibration to also encompass the undeformed equilibrium configuration for a non-trivial duration of the long thermal transient due to the pure thermal conduction. Figure 4 shows the two main phases of the plate deflection during the overall transient (Fig. 4a), which persists until pure thermal conduction occurs [7] (Fig. 4b).



(a) (b)

Figure 3. Initial transient time history (a) and phase portrait (b) of plate deflection W : (red line) only mechanical forcing; (black line) mechanical forcing plus heat impulse.



(a) (b)

Figure 4. Overall transient time histories of plate deflection W and membrane temperature T_{R0}

Conclusions

Thermomechanically coupled, geometrically nonlinear, laminated plates have been addressed through a unified 2D formulation that integrates mechanical and thermal aspects and consistently accounts for cubic variations along the thickness of both the in-plane displacement components and the temperature. Though still constraining the spatial shape of the unknown temperature field along the thickness with respect to the more involved variation patterns possibly observable for non-thin laminates in practical situations, such an assumed consistency of mechanical and thermal variables entails meaningful effects in terms of potential of analysis of the ensuing thermomechanical model. Indeed, it allows us to consider a variety of thermal conditions to be possibly prescribed on the upper and lower surfaces of the laminated plate to represent cases of technical interests (one of which, of mixed nature, has been presented as a numerical case study). This is in contrast with the sole condition of free heat exchange to be possibly taken into account by the CTC model, based on the analogously consistent prescription of a linear variation of the two fields in the thickness direction.

It has also been shown how, under proper assumptions, it is still possible to end up with a minimal ROM (exhibiting a different degree of mechanical-thermal coupling, depending on the thermal conditions prescribed on the plate external surfaces) also in the presence of shear deformability.

Numerical analysis of a case study has provided first hints on the model potential for the analysis of thermomechanical coupling effects in the nonlinear response. The model is expected to be effective for nonlinear dynamic analyses under a variety of mechanical and thermal excitations and/or boundary conditions, and may allow to detect the most important thermomechanical phenomena and their variable importance depending on various thermal properties of the material.

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