DYNAMICS OF PROBABILITY DISTRIBUTION OF STATES OF SEMI-MARKOV SYSTEMS

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Modern technologies of building models of complex systems functioning are based on the use of Markov theory [1]. The corresponding models constructively exploit the Markov property of the behavior of such systems. The simplicity and efficiency of such models are a natural consequence of the fact that in Markov systems the distribution density of the duration of the system's stay in a particular state is determined only by this state but does not depend on when the system got to this state. The non-fulfillment of the Markov property leads to a significant complication of the problem of analyzing the behavior of systems, especially multichannel systems. Difficulties in solving the corresponding problems make the problem of developing methods for constructing models of the behavior of non-Markovian systems urgent.

A semi-Markov process is known to differ from a Markov process in that the distribution laws of the random durations in each of the possible states are not exponential. There are several alternative ways to specify a semi-Markov process. The least demanding of them from the point of view of the amount of information used is as follows. We define sets of possible states E and transitions between them, and a matrix $(Q_{ij}(t))$ of independent distribution functions of the time distribution of the process staying in state i before moving to state j, $i \in E$, $j \in E$. Thus, if t_{ij} – is a random dwell time in i before transitioning to state j, then

$$Q_{ij}(t) = P(t_{ij} < t).$$

Then the transition probability $P_{ij}(t)$ from i to j is the probability that no transition to any other state will occur during this time. This probability is equal to

$$P_{ij}(t) = \int_0^t \prod_{k \neq j} (1 - Q_{ik}(\mathrm{E})) dQ_{ij}(\tau) = P(\zeta(t) = j, t_{ij} < t / \zeta(0) = i), i \neq j.$$

A combination of functions $P_{ij}(t)$ together with the initial state uniquely defines a semi-Markov process.

Let us consider the technology of finding the dynamics of the probability distribution of states of a semi-Markov system as applied to the solution of the simplest problem of reliability theory. The system, which at the initial moment was in state i, can at the moment t be in state j in the following way. First, if $j=i$, the system may not leave state i until time t, or, having left this state, return to it by time t. Second, if $j \neq i$, the system can end up in this state by moving to some intermediate state k at some moment $\tau<t$. In this case

$$G_{ij}(t)=\gamma_i(t)+\sum_{\substack{k\in E\\ k\neq i}} P_{ik}\int_o^t f_{ik}(\tau)\,\mathrm{G}_{kj}(t-\tau)\,\mathrm{d}\,\tau.$$

Here P_{ik} is the probability of transition of the system from state i to state j.

The received system of integral equations is solved using Laplace transformations. The described computational procedure successfully solves the problem. Note, however, that this approach has a fundamental disadvantage consisting of the following. In almost all cases, this solution is a set of functions with numerical coefficients whose values are in no way related to the values of the system parameters. It provides only a point estimate of the system state for any given set of initial data. However, in many cases, for example, when solving control problems of semi-Markov systems, it is necessary to have an analytical description of the dynamics of state probabilities. In addition, it is important to know the dependences of the values of the components of the probability distribution of the system states on the numerical values of the system parameters.

To solve the problem of analyzing semi-Markov systems, we propose an approach based on a special approximation of real processes occurring in the system. Such an approximation should satisfy the following requirements. First, the functions implementing it should be parametrized, i.e., by proper choice of their parameters, it is possible to provide the necessary accuracy of descriptions of real processes occurring in the system. Secondly, the approximating functions should allow simple performance of the forward and inverse Laplace transform.

It is convenient to choose Erlang distribution laws of the required order as such functions. The corresponding functions have a number of important advantages: they are positive on $[0,\infty]$ and integrable; changing the parameters of the Erlang distribution density allows us to change the mathematical expectation, variance, skewness, and kurtosis of the corresponding random variable within a wide range. However, the crucial advantage of Erlang distributions is that the event stream described by this distribution is a sifted Poisson stream. In particular, if we sift the Poisson stream of events by isolating every n event from it, then the random interval between these events will be described by an Erlang distribution of order n. This crucial property of the Erlang flow to be generated by the Poisson flow makes it possible to use it constructively to analyze semi-Markov models almost independently of the type of probability distributions of the real system. The corresponding methodology is two steps.

In the first stage, the distributions describing the incoming flow of failures and the flow of recoveries of the real system are independently approximated by Erlang distributions of proper order. In this case, histograms of the corresponding random variables are formed in a standard way by preliminary processing of the initial data on the duration of stay of the system in the state of normal operation before failure and recovery duration. These histograms are used to estimate the parameters of approximation of Erlang distributions by the maximum likelihood method. In the second stage, the obtained pair of Erlang distributions is used to construct a Markov approximation of the real semi-Markov system. Further investigation is performed by the known methods of Markov theory.

References

1. Dynkin E.B. Markov processes. – M.: Fizmatlit, 1963. -860p.