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SEMI-MARKOV MODELS OF SYSTEM RELIABILITY

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Elementary Models of System Reliability are typically described using Markov theory. However, it is well-known that the distribution laws governing the duration of real systems states before transitioning to another state are not exponential. This fact highlights the importance of developing a method for analyzing such semi-Markov systems.

To address this problem, an approach based on forming and solving a system of integral equations is proposed. During operation, the system can be in one of two states:

E_0 – the state in which the system is functioning normally.

E_1 – the state in which the system has failed and is undergoing repair.

To model these random failure and repair processes, we introduce:

$f_{01}(t)$ – the probability density function for the random duration the system stays in state E_0 before transitioning to state E_1;

$f_{10}(t)$ – the probability density function for the random duration the system stays in state E_1 before transitioning to state E_0;

$H_{00}(t)$ – the conditional probability that the system will be in state E_0 at time t, given that it started in state E_0;

$H_{01}(t)$ – the conditional probability that the system will be in state E_0 at time t, given that it started in state E_0;

$H_{10}(t)$ – the conditional probability that the system will be in state E_0 at time t, given that it started in state E_1;

$H_{11}(t)$ – the conditional probability that the system will be in state E_1 at time t, given that it started in state E_1.

These definitions lead to a set of relationships describing the potential dynamics of the system's states over time.

Let $E=(E_i)$, $i=1, 2,\dots,n$ – represent the set of possible states of the system.

A system initially in state i can be in state j at time t in the following ways. Firstly, if $j=i$, the system may remain in state i, continuously until time t. Alternatively, the system could leave that state, transition through other states, and return to state i by time t. The corresponding mathematical model for this scenario is expressed as:

$$G_{ij}(t) = \gamma_i(t) + \sum_{\substack{k\in E \\ k\neq i}} P_{ik} \int_0^t f_{ik}(\tau) G_{ki}(t-\tau)\,d\tau. \tag{1}$$

Secondly, if $j\neq i$, the system can reach that state by transitioning at some time $\tau < t$ to an intermediate state k. In this case, the dynamics involve:

$$G_{ij}(t) = \gamma_i(t) + \sum_{\substack{k\in E \\ k\neq i}} P_{ik} \int_0^t f_{ik}(\tau) G_{kj}(t-\tau)\,d\tau \tag{2}$$

here, P_{ik} – is the probability of the system transitioning from state i to state j.

In the reliability theory problem under consideration, when the set of states is $E= (E_0, E_1)$, the general relationships (1), (2) are simplified to:

$$H_{00}(t) = \left(1 - \int_0^t f_{01}(\tau)d\tau\right) + \int_o^t f_{01}(\tau)H_{10}(t-\tau)d\tau, \tag{3}$$

$$H_{01}(t) = \int_o^t f_{01}(\tau)H_{11}(t-\tau)d\tau, \tag{4}$$

$$H_{10}(t) = \int_o^t f_{10}(\tau)H_{00}(t-\tau)d\tau, \tag{5}$$

$$H_{00}(t) = \left(1 - \int_0^t f_{10}(\tau)d\tau\right) + \int_o^t f_{10}(\tau)H_{01}(t-\tau)d\tau. \tag{6}$$

The resulting system of integral equations (3)-(6) is solved using Laplace transforms. As is well known, the Laplace transform of the function $u(t)$ is given by the function

$$L(u(t)) = \int_0^\infty u(t)e^{-st}dt = \frac{1}{s}L(u(t)) = \frac{1}{s}u^*(s). \tag{7}$$

Applying the transformation (7) to the relationships (3)-(6), we obtain their Laplace transforms:

$$H^*_{00}(s) = \frac{1}{s}\left(1 - f^*_{01}(s)\right) + f^*_{01}(s)H^*_{10}(s), \tag{8}$$

$$H^*_{01}(s) = f^*_{01}(s)H^*_{11}(s), \tag{9}$$

$$H^*_{10}(s) = f^*_{10}(s)H^*_{00}(s), \tag{10}$$

$$H^*_{11}(s) = \frac{1}{s}\left(1 - f^*_{10}(s)\right) + f^*_{10}(s)H^*_{01}(s). \tag{11}$$

The obtained equations need to be solved by expressing the unknown functions $H^*_{00}(s)$, $H^*_{01}(s)$, $H^*_{10}(s)$, $H^*_{11}(s)$ in terms of the Laplace transforms of the known densities $f_{01}(t)$, $f_{10}(t)$.

For this purpose, by substituting relationships (10) into (8), we obtain

$$H^*_{00}(s) = \frac{1}{s}\left(1 - f^*_{01}(s)\right) + f^*_{01}(s)f^*_{10}(s)H^*_{00}(s),$$

from which we derive

$$H^*_{00}(s)(1 - f^*_{01}(s)f^*_{10}(s)) = \frac{1}{s}(1 - f^*_{01}(s)),$$

$$H^*_{00}(s) = \frac{1}{s}\frac{1 - f^*_{01}(s)}{1 - f^*_{01}(s)f^*_{10}(s)}, H^*_{10}(s) = \frac{1}{s}\frac{1 - f^*_{01}(s))f^*_{10}(s)}{1 - f^*_{01}(s)f^*_{10}(s)},$$

$$H^*_{11}(s) = \frac{1}{s}\frac{1 - f^*_{10}(s)}{1 - f^*_{01}(s)f^*_{10}(s)}, H^*_{01}(s) = \frac{1}{s}\frac{1 - f^*_{10}(s))f^*_{01}(s)}{1 - f^*_{01}(s)f^*_{10}(s)}.$$

Thus, the solution to the problem of analyzing the reliability of the system is reduced to the following two-step procedure. In the first step, it is necessary to obtain the Laplace transforms $f_{01}(s)$, $f_{10}(s)$ of the known densities $f_{01}(t)$, $f_{10}(t)$. In the second step, using the inverse Laplace transform, we obtain the sought functions $H^*_{00}(t)$, $H^*_{01}(t)$, $H^*_{10}(t)$, $H^*_{11}(t)$, which describe the dynamics of the system's states.