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Application of the R-Functions Method for Nonlinear Bending of Orthotropic Shallow Shells on an Elastic Foundation

Lidiya Kurpa[1], Katherine Lyubitska[2*]

Abstract
Geometrically nonlinear behavior of orthotropic shallow shells subjected to the transverse load and resting on Winkler's foundation is investigated On base of the R-function theory and variational methods problem's solution for shells with complex plan form is proposed The algorithm to finding upper and lower critical loads is developed The stress-strain state of shallow shells with the complex planform is investigated including different boundary conditions, properties of material and elastic foundation.

Keywords
R-functions; shell; orthotropic; nonlinear; bending

[1,2] NTU "KhPI", Kharkiv, Ukraine
* **Corresponding author:** lyubitska@mail.ua

Introduction

Composite plates and shells are widely used in many fields: aeronautics, naval industry, building of the space and underwater vehicles, storage tanks, etc. Usually many modern engineering constructions are subjected to a high mechanical loading. As a result, the elements of such designs get larger deflections. Therefore mathematical models of the plates or shells in this case are nonlinear ones. The present study is focused on nonlinear bending and post-buckling analyses for orthotropic shallow shells subjected to transverse loading and resting on elastic foundations. In general case solving of this problem is connected with large mathematical difficulties. To solve these problems numerical methods are used. Among developed methods are: finite differences (FDM), boundary element (BEM), finite element methods (FEM), etc. Many studies have been devoted to analysis of nonlinear behavior of plates and shallow shells [1-10 etc.]. For example, in [1] the Fourier series method was used for static and dynamical analysis of orthotropic plates. Pre-buckling and postbuckling behavior of isotropic shallow spherical shells under uniform transverse load was considered in [4] using "setup" method, by BEM for shear deformable theory in [5]; using Fourier series and the Bessel functions in [6]; by finite-degree Chebyshev polynomials and implicit Houbolt time-marching techniques in [7]. It should be noted that number of works devoted to static and dynamical geometrically nonlinear analysis of anisotropic panels on elastic foundation is not high. Effect of elastic foundation parameters are analyzed in Ref. [2, 3]. Asymptotic iteration method (AIM) is employed to solve the buckling problem for imperfect isotropic shallow spherical shells on Pasternak foundation and nonlinear Winkler foundation. The effect of Winkler-Pasternak foundation on the load-deflection relationship was also investigated in [8, 9, 10 etc.]. The influence of linear and nonlinear Winkler foundation on the average deflection response was studied in [11].

Based on the available papers analysis it follows that there approaches exist which may be applied to shells of circle or rectangular planform with homogeneous boundary conditions. In the present work to solve the system of governing equations the R-functions theory [1], variational methods and the method of incremental loading are used. A principal advantage of this approach is the possibility to investigate orthotropic shallow shells of an arbitrary planform with different boundary conditions and resting on a Winkler-type foundation

1. Mathematical Statement

Considered thin orthotropic shells are assumed to have a relatively small rise as compared to their spans. The shell is subjected to the transverse load $q(x,y)$ and resting on a Winkler-type foundation with modulus p. The fundamental equations for thin shallow shells are obtained using the von Karman theory. Governing equations for large deflections of shallow shells on the base of this theory are:

$$L_2(A_{ij})\varphi + \Delta_k w + \frac{1}{2}L(w,w) = 0 \tag{1}$$

$$L_1(D_{ij})w - \Delta_k \varphi - L(w,\varphi) = q - pw \tag{2}$$

where L, L_1, L_2 and Δ_k are the differential operators defined as:

$$\Delta_k = k_2(\ )_{,xx} + k_1(\ )_{,yy}$$

$$L_1(\ ) = D_{11}(\ )_{,xxxx} + 2(D_{12} + 2D_{66})(\ )_{,xxyy} + D_{22}(\ )_{,yyyy}$$

$$L_2(\ ) = A_{22}(\ )_{,xxxx} + (2A_{12} + A_{66})(\ )_{,xxyy} + A_{11}(\ )_{,yyyy}$$

$$L(w,\varphi) = \varphi_{,xx} \cdot w_{,yy} + \varphi_{,yy} \cdot w_{,xx} - 2\varphi_{,xy} \cdot w_{,xy}$$

Here $w(x,y)$ is the deflection function, $\varphi(x,y)$ is the stress function, k_1, k_2 are the principal curvatures of the panel, $A_{ij} = \dfrac{a_{ij}}{h}$ are elastic constants, $D_{ij}, (ij = 11,22,12,66)$ are stiffness coefficients.

The equilibrium equations system (1)-(2) is supplemented by corresponding boundary conditions. Let's indicate some types of boundary conditions, which will be used below for problem solving:

a) movable clamped edge

$$w = 0,\ w_{,n} = 0,\ \varphi = 0,\ \varphi_{,n} = 0 \tag{3}$$

b) movable simply supported edge

$$w = 0,\ M_n = 0,\ \varphi = 0,\ \varphi_{,n} = 0 \tag{4}$$

where n and τ are normal and tangent to the domain boundary.

2. Solution Method

An analytical solution of the nonlinear system (1) – (2) is impossible in general case. Therefore, different methods of linearization are used. One of the well-known approaches is the method of incremental loading, which was proposed by Vlasov and developed by his followers [12]. Due to the method of incremental loading the given load is divided into n parts $\delta q^{(r)}$ (r=1,2,...n), corresponding to small deflection of a shallow shell. Deflection increment $\delta w^{(r)}$ and the stress function increment $\delta\varphi^{(r)}$ are defined on each loading step. On every r-th step the linearized system is solved by variational Ritz's method. Thus, the problem is reduced to finding of a stationary point of the corresponding functional.

$$\Pi(\delta w^{(r)}, \delta\varphi^{(r)}) = \frac{1}{2}\iint_\Omega \Big\{ D_{11}\left(\delta w^{(r)}_{,xx}\right)^2 + D_{22}\left(\delta w^{(r)}_{,yy}\right)^2 + 2D_{12}\delta w^{(r)}{}_{,xx}\delta w^{(r)}{}_{,yy} + 4D_{66}\left(\delta w^{(r)}_{,xy}\right)^2 +$$

$$2k_1\delta w^{(r)}{}_{,y}\delta\varphi^{(r)}{}_{,y} + 2k_2\delta w^{(r)}{}_{,x}\delta\varphi^{(r)}{}_{,x} - A_{22}\left(\delta\varphi^{(r)}_{,xx}\right)^2 - A_{12}\delta\varphi^{(r)}{}_{,xx}\delta\varphi^{(r)}{}_{,yy} - A_{66}\left(\delta\varphi^{(r)}_{,xy}\right)^2 - A_{11}\left(\delta\varphi^{(r)}_{,yy}\right)^2 +$$

$$+\left(\delta w_{,x}^{(r)}\right)^2\varphi_{,yy}^{(r-1)}-2\delta w^{(r)}{}_{,x}\delta w^{(r)}{}_{,y}\varphi_{,xy}^{(r-1)}+\left(\delta w_{,y}^{(r)}\right)^2\varphi_{,xx}^{(r-1)}+2\left(\delta w^{(r)}{}_{,y}\delta\varphi^{(r)}{}_{,y}w_{,xx}^{(r-1)}+\delta w^{(r)}{}_{,x}\delta\varphi^{(r)}{}_{,x}w_{,yy}^{(r-1)}\right)-$$
$$-2w_{,xy}^{(r-1)}\left(\delta w^{(r)}{}_{,x}\delta\varphi^{(r)}{}_{,y}+\delta w^{(r)}{}_{,y}\delta\varphi^{(r)}{}_{,x}\right)+k\left(\delta w^{(r)}\right)^2-2\delta q^{(r)}\delta w^{(r)}\Big\}d\Omega\,. \quad (5)$$

Here $w^{(r-1)}$ and $\varphi^{(r-1)}$ are functions corresponding to the loading values on the (r-1)-th loading step. According to the Ritz method unknown functions are represented as:

$$\delta w^{(r)}=\sum_{i=1}^{N_1}c_iw_i\,,\quad \delta\varphi^{(r)}=\sum_{N_1+1}^{N_2}c_i\varphi_i$$

were c_i are unknown coefficients, $\{w_i\}$, $\{\varphi_i\}$ are complete sets of coordinate functions, satisfying, at least, kinematic boundary conditions. In the present work construction of such sets for shells with a complex plan form was carried out by the R-functions theory (RFM) [13].

For example, if edge $\delta\omega_1$ of the shallow shell is movable clamped and remaining part of the border is movable simply supported, then the basic functions have the following view:

$$w_i=\omega\omega_1\chi_i \qquad \varphi_i=\omega^2\psi_j. \quad (6)$$

Here $\{\chi_i\}$, $\{\psi_i\}$ are some complete systems of the functions, for examples, power polynomials. The function ω satisfies the next conditions:

$$\omega(x,y)>0\ \forall\ (x,y)\in\Omega\ ;\ \ \omega(x,y)=0\ \ \forall\ (x,y)\in\partial\Omega;\ \ \omega(x,y)<0\ \forall\ (x,y)\notin\Omega$$

The total deflection and stress function are finding as sum of the obtained increments:

$$w^{(n)}=\sum_{r=1}^{n}\delta w^{(r)}\,,\ \varphi^{(n)}=\sum_{r=1}^{n}\delta\varphi^{(r)} \quad (7)$$

To improve the received solution the Newton's method is applied. More detailed description of this method is presented in Ref. [12, 15]. Considering the shells problems there buckling phenomena is possible. Unfortunately, the step-by-step and Newton's methods do not allow to define upper and lower buckling loads of a shallow shell. An approach of building the deformation curve with consecutive passing through all its critical points in conjunction with the RFM was used in [16] for isotropic and orthotropic shallow shells. In the present work the method is developed for orthotropic shells on an elastic foundation. After the variation-structural method application we obtain a system of linear algebraic equations of the following view:

$$\sum_{j=1}^{N_2}a_{ij}c_j-b_i=0\,,\,i=\overline{1,N_2} \quad (8)$$

where variables c_j are the coefficients of indefinite components of the structural formulas. Introduce an additional variable (unknown)

$$c_{N_2+1}=Q=\alpha Q_0$$

Here parameter α is the unknown constant, Q_0 is the function which is defined by the given way of loading.

As system (8) is linear one the next representation of the solution is possible:

$$c_j=\alpha\xi_j \qquad \sum_{j=1}^{N_2}a_{ij}\xi_j-b_j=0 \quad (9)$$

The iterative procedure is used to find the solution of the system (8) at (N_2+1)-dimensional space. Denote $M_1(z_{1j})$ and $M_2(z_{2j})$ the points on the deformation curve corresponding to the two loading steps. The trivial solution $c_{1j}=0$ can be taken as the first points, the solution of linear bending problem c_{2j} can be taken as the second point. Then in (N_2+1) space through M_1 and M_2 we can build the straight line and denote the point $M \in M_1M_2$ with coordinates:

$$c_j = c_{1j} + \lambda(c_{2j} - c_{1j}), \quad j = 1,2,\cdots N_2+1$$

Parameter $\lambda(1<\lambda<3)$ is specially selected [16].

The equation of the hyper-plane, which is perpendicular to the straight line M_1M_2 and passes through the point M is:

$$\sum_{j=1}^{N_2+1}(c_{2j} - c_{1j})[c_j - c_{1j} - \lambda(c_{2j} - c_{1j})] = 0 \tag{10}$$

Substituting (9) into (10) we define the load parameter α :

$$\alpha = \frac{\sum_{j=1}^{N_2+1}(c_{2j} - c_{1j})[c_{1j} - \lambda(c_{2j} - c_{1j})]}{\sum_{j=1}^{N_2+1}(c_{2j} - c_{1j})\xi_j}, \quad \xi_{N_2+1} = 1$$

Using of the proposed approach an extensive computing experiment based on the programming complex POLE-RL [13] has been fulfilled. Some examples are given below to illustrate effectiveness of the method.

3. Numerical Results

Problem 1. Consider a thin isotropic spherical shallow shell based on a circle plane of radius R=0.1(m) resting on elastic Winkler's foundation. Panel is subjected to a uniformly distributed load q. Geometrical and mechanical parameters are the following: $E = 20.4$ (MPa), ν=0.3, $k_1=k_2=4(\mathrm{m}^{-1})$, $k = 4.7\ kH/M^3$, $h=0.01$(m). Shell is assumed to be immovable clamped.

Load-deflection dependence is shown in figure 2. It can be seen that the presented data in Ref. [17] differs from the results obtained by RFM no more than 0.6%. This fact indicates that developed algorithm is valid.

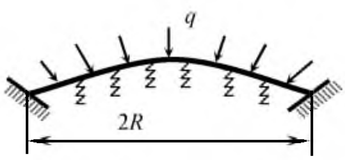

Figure 1. Geometry of a shell

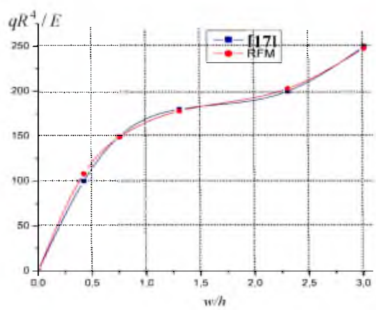

Figure 2. Load-deflection curve

Problem 2. Let's investigate a geometrically nonlinear bending of orthotropic spherical shallow shell with a planform shown in figure 3 under uniformly distributed pressure. Geometrical parameters are the following: h=1, b=0.5a, d=-0.5a. The shell is made of glass-fiber plastic (E_1/E_2=10,

$G_{12}/E_2=1/3$, $\nu_1=0.22$) and resting on elastic Winkler's foundation. Modulus of a foundation is *p*. External contour is movable simply supported, internal contour is movable clamped.

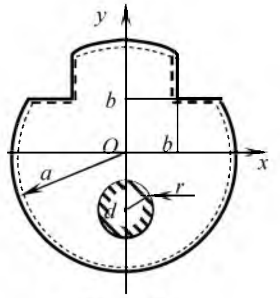

Figure 3.

Effect of foundation on the deformation curve's kind is investigated at $\frac{k_1 a^2}{h}=\frac{k_2 a^2}{h}=30$, $r=0.1a$. Load-deflection dependences are presented in figure 4. After analyzing of obtained results it should be noted that buckling load increases if non-dimensional modulus of the foundation ($\tilde{p}=p\frac{E_2 h^3}{a^4}$) increases. But corresponding maximum deflection w/h increases insignificantly and lies in interval $4< w/h <4.7$. Obviously that difference between upper and lower buckling loads Q_B and Q_H decreases if $\tilde{p}$ increases. And the difference between the maximum deflection corresponding to these loads is reduced. At further growth of the parameter $\tilde{p}$, load-deflection dependence becomes univocal.

The effect of hole's size on critical loads was studied. At the same time a foundation modulus was fixed as $\tilde{p}=10$. The curvature of the spherical shell was varied. The results are shown in figure 5. It can be seen that small increase of the hole's radius considerably increases Q_B, Q_H and brings them together.

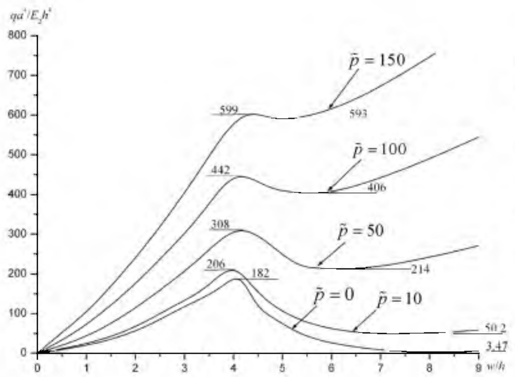

Figure 4. Load-deformation curve

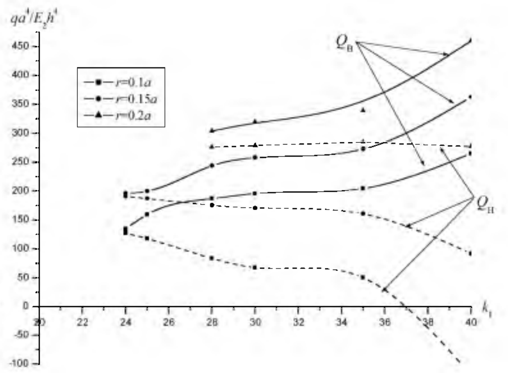

Figure 5. Effect of hole size and curvature of the shell on buckling loads

Effect of the hole size on distribution of the deflection and bending stress in the section y=0 is shown at the table 1. Parameters are put to be $\frac{q_0 a^4}{E_2 h^4}$=100, $\frac{k_1 a^2}{h} = \frac{k_2 a^2}{h}$=10, $\tilde{p}$=20. The table shows that with increasing size of the hole deflections and stresses $\sigma_y^И$ grow, while $\sigma_x^И$ is reduced.

Table 1. Deflection and bending stress at the y=0 section

x	0	0.2	0.4	0.6	0.8	1
	r=0.1a					
$w(0,0)/h$	1.85	1.93	1.95	1.66	1.01	0
$\sigma_x^И a^3 / E_2 h^3$	–1.23	1.38	3.49	3.16	4.66	4.54
$\sigma_y^И a^3 / E_2 h^3$	2.76	2.63	2.14	1.77	1.46	0.75
	r=0.2a					
$w(0,0)/h$	1.16	1.28	1.40	1.25	0.80	0
$\sigma_x^И a^3 / E_2 h^3$	–2.15	0.49	3.24	2.47	4.49	4.20
$\sigma_y^И a^3 / E_2 h^3$	1.58	1.84	1.76	1.40	1.24	0.64
	r=0.3a					
$w(0,0)/h$	0.57	0.71	0.93	0.92	0.63	0
$\sigma_x^И a^3 / E_2 h^3$	–2.65	–0.75	2.77	2.40	4.15	3.34
$\sigma_y^И a^3 / E_2 h^3$	–0.08	0.61	1.29	1.16	1.06	0.51

Table 2 shows the extreme values of deflection, bending and membrane stresses in the shell at $\frac{k_1 a^2}{h} = \frac{k_2 a^2}{h}$=10, r=0.2a. It can be concluded that increasing of the parameter $\tilde{p}$ reduces the tension of a shallow shell what corresponds to the physical meaning of the problem.

Table 2. Effect of $\tilde{p}$ on deflection and stresses of the shell

	w/h max	$\sigma_x^B \frac{a^3}{E_2 h^3}$		$\sigma_y^B \frac{a^3}{E_2 h^3}$		$\sigma_x^M \frac{a^3}{E_2 h^3}$		$\sigma_y^M \frac{a^3}{E_2 h^3}$	
		max	min	max	min	max	min	max	min
$\tilde{p}$=0	2.061	7.046	–13.10	3.347	–3.968	14.109	–9.791	5.856	–6.341
$\tilde{p}$=20	1.581	5.548	–10.43	2.500	–3.093	10.740	–7.409	4.806	–4.995
$\tilde{p}$=40	1.283	4.641	–8.726	1.985	–2.548	8.613	–5.884	4.101	–4.168

Conclusions

This work presents a numerically-analytical method of analysis of the geometrically-nonlinear shallow shells with complex planform. It is assumed that shells are resting on elastic foundation and subjected to transverse load. The proposed approach is based on Ritz's variational method, R-function theory, methods of step-by-step loading and Newton-Kantorovitch's, method of consecutive approximations. Good agreement between the obtained results and the available data shows effectiveness of the developed method. From computation results it follows that the choice of foundation modulus can prevent structures from buckling. Technological cutouts also significantly affect on the behavior of structures. So, variation of geometric and physical characteristics allows to choose the optimal parameters of structural elements and to control their strength.

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