Application of Solution Structure Method to Modeling Dynamic Response of Mechanical Structures

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Abstract

Transient nature of the loading conditions applied to the structural components makes dynamic analysis one of the important components in the design-analysis cycle. Time-varying forces and accelerations can substantially change stress distributions and cause a premature failure of the mechanical structures. In addition, it is also important to determine dynamic response of the structural elements to the frequency of the applied loads. In this paper we describe an application of the meshfree Solution Structure Method to the structural dynamics problems. Solution Structure Method is a meshfree method which enables construction of the solutions to the engineering problems that satisfy exactly *all prescribed boundary conditions. This method is capable of using spatial meshes that do not conform to the shape of a geometric model. Instead of using the grid nodes to enforce boundary conditions, it employs* distance fields *to the geometric boundaries and combines them with the basis functions and prescribed boundary conditions at run time. This defines unprecedented geometric flexibility of the Solution Structure Method as well as the complete automation of the solution procedure.*

Keywords

Dynamic response, natural frequencies, Finite Element Analysis, Solution Structure Method, meshfree method, distance fields.

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Introduction

This paper describes an application of the Solution Structure Method to the structural dynamics problems. This method can be classified as a generalized finite element method which can use a spatial mesh of elements that does not conform to the shape of the geometric model. Meshfree nature of the method makes it possible to break free from the limitation imposed by the standard FEM meshing and achieve much higher flexibility in handling geometry and imposing the prescribed boundary conditions. While other meshfree methods experience difficulties with enforcing the essential boundary conditions, the Solution Structure Method offers a mathematically elegant approach that incorporates the prescribed boundary conditions into the solution on the analytical level. This method was originated by Kantorovich [1] and later generalized by Rvachev and his students [2–4]. The method is based on a simple idea to represent the solution to a boundary value problem by solution structure — a power series of the functions that vanish on the geometric boundaries. If these functions resemble an mth order approximation of the Euclidean distance to the boundary, this approach enables exact treatment not only essential boundary conditions, but also the natural boundary conditions that include normal derivatives of the solution [2, 3]. The Solution Structure Method is also known under its alternative names — R-function method and meshfree method with distance fields. In this paper we prefer to use the term “Solution Structure Method” — to emphasize the fact that the components of the displacement vector will be represented by the solution structures enabling exact treatment of the essential boundary conditions.

In this paper we will focus mostly on 2D structural dynamics problems, but the proposed approach is readily applied to 3D problems. Therefore, all mathematical derivations in this paper were performed with no connection to the dimension of the underlying space. We will validate and compare the proposed approach with the solutions obtained using the traditional Finite Element Method in ANSYS Workbench.

1. Modal analysis with the Solution Structure Method

In the present paper we will use a weak formulation of a structural dynamics problem that describes the balance of the work done by the applied loads, kinetic and potential energy of the deformed structure [5]:

$$\int_{\Omega} \mathsf{u}^T \rho \ddot{\mathsf{u}} d\Omega + \int_{\Omega} \mathsf{u}^T C \dot{\mathsf{u}} d\Omega + \int_{\Omega} \varepsilon^T \sigma d\Omega - \int_{\Omega} \mathsf{b}^T \mathsf{u} d\Omega - \int_{\partial\Omega_t} \mathsf{q}^T \mathsf{u} dS = 0, \tag{1}$$

where u, $\dot{\mathsf{u}}$ and $\ddot{\mathsf{u}}$ are the displacement, velocity and acceleration vectors; ρ is the specific weight of the material; C is a damping coefficient; ε and σ are the acting strains and stresses; $\mathsf{b} = \mathsf{b}(\mathsf{x},t)$ and $\mathsf{q} = \mathsf{q}(\mathsf{x},t)$ are body and traction forces respectively. Weak formulation (1) requires the displacements u to satisfy the kinematic (essential) boundary conditions:

$$\mathsf{u}|_{\partial\Omega_u} = \overline{\mathsf{u}}, \tag{2}$$

where $\overline{\mathsf{u}}$ are the displacements prescribed on the $\partial\Omega_u$ portion of the geometric boundary. As usual, zero values of $\overline{\mathsf{u}}$ correspond to the fixed boundary. In this paper we will use a bar over the mathematical symbols to denote the specified quantities.

In order to obtain the unique solution to the structural dynamics problem the initial distributions of the displacements and velocities have to be specified:

$$\mathsf{u}|_{t=0} = \overline{\mathsf{u}}_0(\mathsf{x}); \; \dot{\mathsf{u}}|_{t=0} = \overline{\mathsf{V}}_0(\mathsf{x}). \tag{3}$$

To solve structural dynamics problem(1, 2, 3) we will employ a modal analysis approach [5,6]. It represents the time dependent displacement field by linear combination of the natural vibration modes:

$$\mathsf{u}(\mathsf{x},t) = \sum_{i=1}^{m} \tilde{\mathsf{u}}_i(\mathsf{x}) U_i(t) + \overline{\mathsf{u}}^*(\mathsf{x},t). \tag{4}$$

In this expression $\tilde{\mathsf{u}}_i(\mathsf{x})$, $i = 1,\ldots,m$ are the free vibration modes, $U_i(t)$ are time-dependent coefficients that define the contribution of each vibration mode at every moment of time. Since natural vibration modes satisfy homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega_u$, to accommodate the prescribed non-zero displacements (2) we have to add an extra term $\overline{\mathsf{u}}^*(\mathsf{x},t)$ that interpolates non-homogeneous displacements and extends them inside the geometric domain.

Natural (resonance) frequencies and vibration modes are the solution of the generalized eigenvalue problem [7,8]:

$$[\mathsf{K}][\tilde{\mathsf{u}}] = \lambda^2 [\mathsf{M}][\tilde{\mathsf{u}}]. \tag{5}$$

Natural modes are represented by the linear combinations of the basis functions $[\eta]$ satisfying homogeneous Dirichlet boundary conditions [7]:

$$\tilde{\mathsf{u}}_i(\mathsf{x}) = [\tilde{\mathsf{u}}_i][\eta]. \tag{6}$$

As we described in [3,7–9] such basis functions can be constructed as a product of the functions ω^x, ω^y, and ω^z that vanish on the fixed boundaries and the basis functions χ_j: $\eta_j = [\omega^x, \omega^y, \omega^z]^T \chi_j$.

Substituting $\tilde{\mathsf{u}}_i$ (6) into expression (4) and then into the weak formulation (1) we obtain the following system of algebraic equations:

$$\sum_{i=1}^{m} \left[[\mathsf{M}][\tilde{\mathsf{u}}_i] \ddot{U}_i(t) + [\mathsf{C}][\tilde{\mathsf{u}}_i] \dot{U}_i(t) + [\mathsf{K}][\tilde{\mathsf{u}}_i] U_i(t) \right] = [\mathsf{F}]. \tag{7}$$

In this system $[\mathsf{K}]$, $[\mathsf{M}]$ and $[\mathsf{C}]$ are stiffness, mass and damping matrices.

Using normal modes to represent solution of a structural dynamics problem provides an easy way to separate spatial and temporal variables. In addition, orthogonality of the normal modes makes it possible to reduce the system (7) to a system of ordinary differential equations with respect to $U_i(t)$:

$$\ddot{U}_i(t) + 2\lambda_i \varepsilon_i \dot{U}_i(t) + \lambda_i^2 U_i(t) = f_i(t), \quad i = 1,\ldots,m, \tag{8}$$

where $f_i(t)$ is given as a dot product of the load vector $[\mathsf{F}]$ and ith eigen vector of the generalized eigenvalue problem (5):

$$f_i(t) = [\bar{\mathsf{u}}_i]^T [\mathsf{F}]. \tag{9}$$

The coefficient ε_i in the equation (8) represents the Rayleigh damping for which $\varepsilon_i = \frac{\alpha}{2\lambda_i} + \frac{\beta\lambda_i}{2}$. Since the damping is difficult to measure or obtain directly, the damping is usually assumed and parametric studies are performed. The order of the differential equations in (8) can be further reduced if we add velocity V_i as an additional variable:

$$\begin{cases} \dot{U}_i = V_i, \\ \dot{V}_i = f_i(t) - \lambda_i^2 U_i(t) - 2\lambda_i\varepsilon_i V_i(t) \end{cases}. \tag{10}$$

At this point the solution process of the structural dynamics problem becomes straightforward: first by solving (5) the resonance frequencies and normal modes are computed. This involves assembly of the stiffness and mass matrices and solution of the generalized eigenvalue problem [7]. This requires substantial computational resources, but, fortunately, for linear problems, this is one time expense. Then, temporal solutions are obtained by solving (10) for each normal mode. This involves computation of the load vector $[\mathsf{F}]$ at every time instance and application of an appropriate ODE solution method. Because of its simplicity of implementation, in this paper we will use an implicit trapezoidal method.

1.1 Treatment of the boundary conditions using Solution Structure Method

The essence of the Solution Structure Method is the ability to construct the solutions to the engineering analysis problems providing the exact treatment of the prescribed boundary conditions. The main idea of the method is to represent a solution to the boundary value problem by powers of some function ω that vanishes on the boundary of a geometric domain:

$$u(\omega) = u(0) + \sum_{k=1}^{m} \frac{1}{k!} u_k(0)\omega^k + \omega^{k+1}\Phi. \tag{11}$$

As we demonstrated in [3] this representation is a generalization of a classical Taylor series. Coefficients $u_k(0)$ represent the kth order normal derivatives of the solution that are prescribed on the geometric boundary $\partial\Omega$. The remainder term $\omega^{k+1}\Phi$ assures completeness, and is usually used to satisfy the governing equation(s) of the problem.

In this paper we will use the Solution Structure Method to satisfy the kinematic (essential) boundary conditions. Let us consider two possibilities: (1) when the boundary is fixed, and (2) when some non-zero values of the displacement are specified on the geometric boundaries. For simplicity we will explain the treatment of the kinematic boundary conditions just for one component of the displacement vector since the same approach can be used for the other two components.

In the case of the fixed boundaries, zero displacements can be enforced by a simple solution structure which was proposed by Kantorovich in [1]:

$$u_x = \omega^x\Phi. \tag{12}$$

In this expression ω^x is a function that vanishes on the fixed boundary $\partial\Omega_{u_x}$. Such functions can be constructed by a variety of the methods. Constructive methods [10] use the standard geometric representations and translate them into the real-valued functions that takes on zero value on the boundary. Among constructive methods it would be worth mentioning the use of Ricci functions [11], R-functions [2, 12, 13], and function-based modeling approach [14, 15].

The constructed distance fields to the essential boundaries are combined with a set of the basis functions that can be specified over a spatial grid that does not conform to the shape of the geometric model. In this case the distance fields $\omega^{x,y,z}$ enforce zero displacements in each coordinate direction at the points where $\omega^{x,y,z}$ vanish.

In the case when the non-homogeneous displacements are prescribed at different portions of the geometric boundary, components of the displacement vector can be represented by a slightly modified solution structure:

$$u_{x,y,z} = \omega^{x,y,z}\Phi + \overline{u}^{*}_{x,y,z}, \tag{13}$$

where the functions $\overline{u}^*_{x,y,z}$ interpolate the prescribed displacements in x, y and z directions. Transfinite interpolation technique, which is a generalization of an inverse weighting distance interpolation [16], combines the approximate distance fields $\omega_i^{x,y,z}$ to the geometric boundaries with the prescribed values $\overline{u}_{(x,y,z)_i}$ of the displacements [17]:

$$\overline{u}^*_{x,y,z} = \sum_{i=1}^{k} \overline{u}_{(x,y,z)_i} \frac{\prod_{j=1;j\neq i}^{k} \omega_j^{x,y,z}}{\sum_{l=1}^{k} \prod_{j=1;j\neq l}^{k} \omega_j^{x,y,z}}. \tag{14}$$

Expression (14) also extends the specified displacements inside the geometric domain, which makes it possible to use $\overline{u}^*_{x,y,z}$ to compute the load vector $[\mathsf{F}]$.

1.2 Treatment of the initial conditions

As we discussed earlier, in most cases the solution of structural dynamic problems requires specification of the initial distributions of the displacements and velocities (3). When modal analysis is used, the representation of the solution by the expression (4) necessitates the initial conditions to be re-represented by linear combinations of the natural vibration modes:

$$\overline{\mathsf{u}}_0(\mathsf{x}) \approx \sum_{i=1}^{m} U_i^0 \tilde{\mathsf{u}}_i(\mathsf{x}) + \overline{\mathsf{u}}^*(\mathsf{x},t)|_{t=0}, \tag{15}$$

$$\overline{\mathsf{V}}_0(\mathsf{x}) \approx \sum_{i=1}^{m} V_i^0 \tilde{\mathsf{u}}_i(\mathsf{x}) + \dot{\mathsf{u}}^*(\mathsf{x},t)\Big|_{t=0}. \tag{16}$$

Since the normal modes $\tilde{\mathsf{u}}_i$ are orthogonal to each other, it does not take much effort to compute the coefficients U_j^0 from the expression (15) using least square fit:

$$U_j^0 = [\mathsf{F_u}]^T [\tilde{\mathsf{u}}_j], \qquad j = 1,\ldots,m, \tag{17}$$

where the elements of the vector $[\mathsf{F_u}]$ are given by the expression:

$$F_{u_i} = \int_{\Omega} (\overline{\mathsf{u}}_0(\mathsf{x}) - \overline{\mathsf{u}}^*(\mathsf{x},t)|_{t=0}) \rho \eta_i d\Omega. \tag{18}$$

Similarly we can obtain the coefficients V_j^0 for the velocity field:

$$V_j^0 = [\mathsf{F_v}]^T [\tilde{\mathsf{u}}_j], \qquad j = 1,\ldots,m, \tag{19}$$

$$F_{v_i} = \int_{\Omega} \left(\overline{\mathsf{V}}_0(\mathsf{x}) - \dot{\mathsf{u}}^*(\mathsf{x},t)\Big|_{t=0}\right) \rho \eta_i d\Omega. \tag{20}$$

In the case when the motion starts from a statically deformed state, all V_j^0 have to be zero, but U_j^0 can be determined from the static equilibrium equation:

$$\sum_{i=1}^{m} [\mathsf{K}] [\tilde{\mathsf{u}}_i] U_i^0 = [\mathsf{F}], \tag{21}$$

where the elements of load vector $[\mathsf{F}]$ is given by the following expression:

$$F_i = -\int_{\Omega} \mathsf{B}^T [\eta_i] \mathsf{D} \mathsf{B} [\overline{\mathsf{u}}^*] d\Omega + \int_{\Omega} \mathsf{b}^T|_{t=0} \eta_i d\Omega + \int_{\partial\Omega_t} \mathsf{q}^T|_{t=0} \eta_i dS.$$

Using orthogonality of the normal modes the equation (21) can be transformed into expression that gives numerical values to the coefficients U_j^0:

$$U_j^0 = \frac{[\tilde{\mathsf{u}}_j]\,[\mathsf{F}]}{\lambda_j^2}. \tag{22}$$

If the boundary, on which the displacements are prescribed, is the same for the static (21) and dynamic (7) problems, only a few normal modes are needed in the expression (15) to provide an accurate approximation of the initial displacement. However, in the case when different boundaries were used to prescribe the fixations and non-homogeneous displacements for the initial displacement distribution and for the dynamic problem, many more normal modes are needed in the expression (15) to assure the approximation accuracy.

In many structural dynamics problems the motion starts from a completely undisturbed state. In this case all U_i^0 and V_i^0 have zero values.

2. Numerical experiments and convergence study

2.1 Dynamic response due to non-homogeneous initial conditions

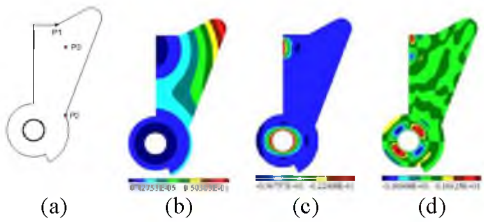

Figure 1. (a) Geometry of a gun hammer with the observation points. (b) Distance field to fixed boundary (shown by a thick line in Fig. 1(a)). Approximations of the initial velocity components obtained using 250 normal vibration modes and Lanczos filtering: (c) $\overline{V}_{0x}$, (d)$\overline{V}_{0y}$.

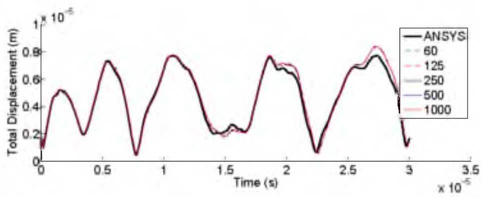

Figure 2. Time history of the magnitude of the displacement at point $P2$ on a gun hammer model (Fig. 1(a)) due to the applied initial velocity.

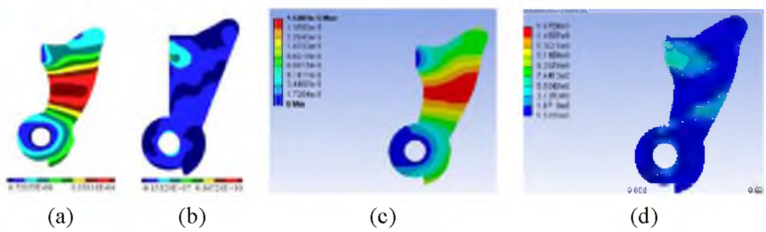

Figure 3. Dynamic structural analysis of a gun hammer in Fig. 1(a): (a) Magnitude of the displacement and (b) von Mises stress at $t = 2.5 \cdot 10^{-5}$ s obtained using meshfree Solution Structure Method with 250 normal modes. Dynamic structural analysis of a gun hammer in Fig. 1(a): (c) Magnitude of the displacement and (d) von Mises stress at $t = 2.5 \cdot 10^{-5}$ s obtained using traditional FEA in ANSYS Workbench.

Accuracy of the solutions of structural dynamics problems, as any other transient problem, also depends on the accuracy of representing the initial conditions. Using free vibration modes as the basis functions to represent the dynamic solution requires the initial conditions to be represented in the same way. In the previous examples we have used the homogeneous initial conditions which describe undeformed and stationary mechanical structures. In many situations motion starts from an undisturbed state, however in some cases we have to deal with non-zero displacements and/or non-zero velocities prescribed as the initial conditions. Usually, if the essential boundaries stay the same at the initial moment of time and during the dynamic simulation, incorporation of nonhomogeneous displacements into the initial conditions is not so difficult. The initial displacement is represented by the same solution structure (4, 15) in which the degrees of freedom U_j^0 are computed using either expression (17) or (22). If the essential boundaries at the initial moment and during the dynamic process are not the same, treatment of the initial conditions becomes more challenging. In this case, the basis functions (normal modes) used to represent the dynamic displacements do not satisfy the essential boundary conditions at the initial moment of time. A similar situation occurs when a nonhomogeneous initial velocity field has to be defined.

Let us consider a simulation of the dynamic behavior of a gun hammer, whose geometry is shown in Fig. 1(a). In order to compare our modeling results with the results produced by the FEA in Ansys Workbench several assumptions and simplifications have been made. First, we assume that the hammer has initial velocity of $V_x = -5m/s$, $V_y = 0m/s$, i.e. all points of the hammer move horizontally with the same velocity. This simplification has been made, because Ansys Workbench does not allow rotational velocity to be specified as the initial condition. In addition, we assume that after the hammer hits the firing pin, the boundary of the circular hole and contact boundary with the firing pin are fixed in both coordinate directions.

In order to proceed with the meshfree simulations, an approximate distance field to the fixed boundaries (Fig. 1(b)) has been constructed from the geometric model in Fig. 1(a). This distance field will be used in the solution structure (13) to enforce zero displacements on the fixed boundaries. To construct the normal modes we use biquadratic tensor product B-splines defined over 100×170 uniform Cartesian grid. To obtain the dynamic solution we used 60, 125, 250, 500, and 1000 eigen modes. Contour plots in Fig. 1(c) and Fig. 1(d) present the approximation of the initial velocity components using 250 eigen modes. To produce these approximations we have used a σ-approximation, but the Gibbs phenomenon is still noticeable.

The numerical integration of the governing equation (8) was performed using implicit trapezoidal integrator with time step of $3\cdot10^{-8}$s. We can observe that the maximum disagreement between the meshfree and finite element solutions occurs at the point $P2$ (Fig. 1(a)). Analyzing the time history plot for this point presented in Fig. 2 we observe a good agreement between meshfree and finite element solutions in the first half of the time interval. The difference between solutions gets larger towards the end of the time interval. This can be explained by the fact that the finite element solution was obtained in Ansys Workbench by the direct integration of the equation of motion (1) which gives accurate results for the relatively short time intervals.

We also compared the magnitude of the displacement and von Mises stress at the moment of time $t = 2.5\cdot10^{-5}$s that corresponds to one of the local maximums of the magnitude of the displacement. Close analysis of the plots in Fig. 3 shows that the results obtained by the meshfree Solution Structure and Finite Element Methods are very close to each other. The difference between the magnitude of the displacements obtained by these methods (Fig. 3(a) and Fig. 3(c)) do not exceed 2.5% measured using C_0 norm. The Solution Structure Method captures the stress singularities at the ends of the fixed line segment better in comparison with the FEA, thus resulting in the maximum von Mises stress value to be 25% more than the one predicted by the traditional FEA. We plotted the von Mises stress distributions using the common scale for the color contour lines. This allows us to compare the stress values at other points. From Fig. 3(b) and Fig. 3(c) we can see that the von Mises stresses obtained by both methods exhibit very similar behavior.

3. Conclusions

In this paper we presented an application of the Solution Structure Method to solution of structural dynamics problems and investigated its convergence properties. Our numerical experiments demonstrated fast convergence of the proposed approach and a very good agreement with modeling results obtained by the traditional Finite Element Analysis.

The proposed method combines advantages of the Solution Structure Method and modal vibration analysis. In particular, using solution structures to represent normal vibration modes makes it possible to exactly satisfy the prescribed kinematic boundary conditions. One of the major advantages of the Solution Structure Method is its ability to use the grids of basis functions that do not conform to the shape of a geometric model. In our examples we used Cartesian grids of tensor product B-splines as the basis functions to represent the free vibration modes. This substantially simplifies creation of spatial meshes of the basis functions for the analysis and reduces the data preparation time.

Mathematically, the modal analysis methods provide a very convenient way to reduce the computational cost of the solution. They make it possible to separate the spatial and temporal variables and further reduce the mathematical complexity of the problem to solving the system of ordinary differential equations, but only for temporal variables [5,6]. Modal analysis, however, requires computation of natural frequencies and vibration modes, but this is a one time expense which will be offset by cheaper computational cost of marching in time.

Modal analysis separates spatial and time variables and transforms the weak formulation (1) into a system of second order ordinary differential equations (8) which can be solved by a variety of methods. In our work we used a trapezoidal time integrator. Being implicit, the trapezoidal method can be used to solve very stiff (with large damping) ordinary differential equations without imposing severe penalty on the choice of the time step. Modal approach makes it possible to reduce the dimension of the approximation space of the problem. For smooth (in time) loads pretty accurate solutions can be obtained using a few normal modes, effectively reducing the computational cost of the solution in comparison with the direct methods. Impulse loads or suddenly applied loads require inclusion of higher frequency normal modes, but even in these cases the number of the degrees of freedom stays smaller than otherwise would be required by the direct solution method. Orthogonality of the normal modes makes it possible to exclude solution of the algebraic systems. Instead, computation of the load vector and dot product (9) of the load and eigen vectors are only necessary.

Numerical experiments, presented in this paper, demonstrated very good convergence of the solutions obtained by the meshfree Solution Structure Method as well as good agreement with the FEA results. We performed side-by-side comparisons of the displacements, velocities and stresses computed by both methods and observed their very good agreement.

When normal modes have been selected to represent the dynamic solution, the frequency of the applied load has to be taken into account. The reason is simple: because each basis function (normal mode) corresponds to some vibration frequency, the normal modes with frequencies that are close to the frequency of the applied load have to be included to ensure the accuracy of the solution. In most cases, when the frequencies of the external loads are known, the normal modes extraction can be performed on the interval that includes the smallest and largest frequencies of the applied loads.

From computational point of view using the modal analysis approach can reduce the amount of memory needed to store the dynamic solution. Let N be a number of degrees of freedom used to approximate the normal modes. If m normal modes are used to construct the dynamic solution, then Nm elements have to be stored to define all normal modes in the expression (4). Progressing in time requires storage of two coefficients U_i and V_i for each normal mode for each time instance. n_t time steps will require storage of $Nm + 2m(n_t + 1)$ of coefficients. In contrast, direct solution of the motion equation (1) will require storage of Nn_t degrees of freedom. When number of the time steps, n_t, is larger than the number of the normal modes, m, used to represent the solution the modal analysis approach requires less memory storage for the complete time history of the solution. In addition, modal analysis approach make it possible to obtain the distribution of the displacements at the intermediate time instances by a simple interpolation of U_i values.

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