

*M. ITSKOV, O.O. LARIN***SIMPLE SHEAR: EIGENVECTORS OF CAUCHY-GREEN TENSORS ROTATE AGAINST EACH OTHER**

Simple shear represents a somewhat complex case of deformation although it is very good studied. In this paper, we discuss a new aspect of simple shear which has not been observed before. Rotations of the eigenvectors of the right and left Cauchy-Green tensors with increasing amount of shear under the kinematically defined simple shear are theoretically studied. An analysis has been done within a framework of the nonlinear theory of elasticity. Mathematical processor Maple is used for the calculations and animation of the results. Phenomena of mutually opposite rotations of the eigenvectors of the right and left Cauchy-Green tensors is found that can be important for anisotropic and in particular fibre-reinforced materials. We studied rotations of principal strain directions under the kinematically defined simple shear. Accordingly, eigenvectors of the right and left Cauchy-Green tensors rotate against each other with the increasing amount of shear. Interestingly, the eigenvectors of  $\mathbf{b}$  rotate in the same direction as line elements of the material while the eigenvectors of  $\mathbf{C}$  in the opposite direction. For example, this can be important for anisotropic and in particular fiber reinforced materials. In this case, the direction of the maximal stretch will rotate with respect to reinforcement directions.

**Keywords:** simple shear, rotation, eigenvectors, Cauchy-Green tensors.

*M. ИЦКОВ, О.О. ЛАРИН***ПРОСТЫЙ ЗСУВ: ВЛАСНІ ВЕКТОРИ ТЕНЗОРІВ КОШІ-ГРИНА ОБЕРТАЮТЬСЯ ОДИН ПРОТИ ІНШОГО**

Простий зсув являє собою дещо складний випадок деформації, хоча він дуже добре вивчений. У цій роботі ми обговорюємо новий аспект простого зсуву, який раніше не спостерігався. Теоретично вивчаються обертання власних векторів правого та лівого тензорів Коші-Гріна зі збільшенням кількості зсуву під кінематично визначеним простим зсувом. Аналіз зроблено в рамках нелінійної теорії пружності. Математичний процесор Maple використовується для обчислень та анімації результатів. З'являються взаємно протилежні обертання власних векторів правого та лівого тензорів Коші-Гріна, які можуть мати важливе значення для анізотропних і зокрема армованих волокнами матеріалів.

**Ключові слова:** простий зсув, обертання, власні вектори, тензори Коші-Гріна.

*M. ИЦКОВ, А.А. ЛАРИН***ПРОСТОЙ СДВИГ: СОБСТВЕННЫЕ ВЕКТОРЫ ТЕНЗОРОВ КОШИ-ГРИНА ВРАЩАЮТСЯ ДРУГ ПРОТИВ ДРУГА**

Простой сдвиг представляет собой довольно сложный случай деформации, хотя он очень хорошо изучен. В этой статье мы обсуждаем новый аспект простого сдвига, который ранее не наблюдался. Теоретически исследованы повороты собственных векторов правого и левого тензоров Коши-Грина с увеличением величины сдвига при кинематически заданном простом сдвиге. Анализ проведен в рамках нелинейной теории упругости. Математический процессор Maple используется для расчетов и анимации результатов. Обнаружены явления взаимно противоположных вращений собственных векторов правого и левого тензоров Коши-Грина, которые могут быть важны для анизотропных и, в частности, армированных волокном материалов.

**Ключевые слова:** простой сдвиг, вращение, собственные векторы, тензоры Коши-Грина.

**Introduction.** Simple shear represents a very important case of deformation which can be defined either in terms of stresses or strains. Both definitions generally lead to different deformation states which coincide only in the reference (undeformed) configuration (see, e.g. [1]). The kinematical (strain based) definition of simple shear was first formulated by Rivlin in 1948 [4]. Since then, the so defined simple shear has intensively been studied. Ogden [3] dealt with Lagrangian and Eulerian orientations of principal strain directions. He expressed the double inclination angles of these directions and obtained the bounds for these angles in the reference configuration and in the

case of infinite shear.

In the present paper, we study rotations of these angles with the increasing amount of shear. It appears that the principal directions of the right and left Cauchy-Green tensors (Lagrangian and Eulerian, respectively) rotate against each other. This interesting aspect of simple shear seems to be overseen in literature.

**Simple share.** Consider a rectangular sheet one edge of which is moved parallel to the opposite one so that the distance between edges remains constant (Fig. 1). Let  $\mathbf{x} = x^i \mathbf{e}_i$  and  $\mathbf{X} = X^i \mathbf{e}_i$  be position vectors of a material

point of the sheet in its current and reference configuration, respectively, where  $e_i = e^i$  ( $i = 1,2,3$ ) represent orthonormal vectors. Assuming that the deformations of the sheet are homogeneous we can write [4]

$$x^1 = X^1 + \gamma \cdot X^2; \quad x^2 = X^2; \quad x^3 = X^3, \quad (1)$$

where  $\gamma = \tan \phi$  is referred to as the amount of shear while  $\phi$  denotes the shear angle. Thus, the deformation gradient  $F = \text{grad } x$  can be given by  $F = F^i_j \cdot e_i \otimes e^j$ , where the matrix  $[F^i_j]$  is represented by

$$[F^i_j] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2)$$

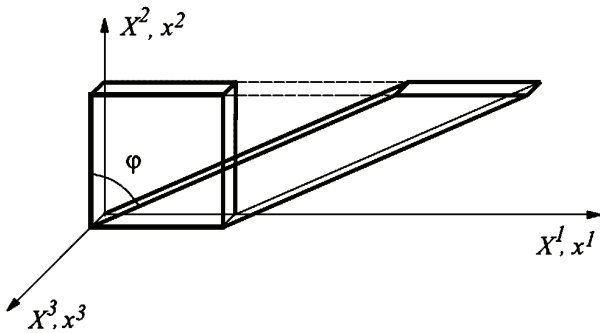


Figure 1 – Simple shear of a rectangular sheet

For the tensor notation and summation convention we refer to [2].

The right and left Cauchy-Green tensors take thus the form

$$C = F^T F = C^i_j e_i \otimes e^j \quad \text{and} \quad b = F F^T = b^i_j e_i \otimes e^j,$$

respectively, where

$$[C^i_j] = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 + \gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad [b^i_j] = \begin{bmatrix} 1 + \gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

The characteristic equation of these matrices can be written by

$$(1 - \Lambda) [(1 - \Lambda) (1 + \gamma^2 - \Lambda) - \gamma^2] = 0. \quad (4)$$

Eigenvalues of  $C$  and  $b$  are roots of (4) which can be expressed by

$$\Lambda_{1/2} = 1 + \frac{\gamma^2 \pm \sqrt{4\gamma^2 + \gamma^4}}{2} = \left( \frac{\sqrt{4 + \gamma^2} \pm \gamma}{2} \right)^2 \quad \Lambda_3 = 1. \quad (5)$$

The eigenvectors  $N = N^i \cdot e_i$  of  $C$  corresponding to the first two eigenvalues result from the equation system:

$$\begin{cases} \frac{-\gamma^2 \mp \sqrt{4\gamma^2 + \gamma^4}}{2} \cdot N^1 + \gamma \cdot N^2 = 0; \\ \gamma \cdot N^1 + \frac{\gamma^2 \mp \sqrt{4\gamma^2 + \gamma^4}}{2} \cdot N^2 = 0; \\ \frac{-\gamma^2 \mp \sqrt{4\gamma^2 + \gamma^4}}{2} \cdot N^2 = 0. \end{cases}$$

Since the first and second equation are equivalent we only obtain

$$N^2 = \frac{\gamma \pm \sqrt{4 + \gamma^2}}{2} \cdot N^1; \quad N^3 = 0.$$

so that  $N^2 = \sqrt{\Lambda_1} \cdot N^1$  or  $N^2 = -\sqrt{\Lambda_2} \cdot N^1$ . After the normalization this yields the following unit eigenvectors

$$N_1 = \frac{1}{\sqrt{1 + \Lambda_1}} \cdot e_1 + \frac{1}{\sqrt{1 + \Lambda_2}} \cdot e_2; \quad (6)$$

$$N_2 = \frac{1}{\sqrt{1 + \Lambda_2}} \cdot e_1 - \frac{1}{\sqrt{1 + \Lambda_1}} \cdot e_2.$$

where we take into account that  $\Lambda_1 \cdot \Lambda_2 = 1$ . Applying the above procedure to the third eigenvector corresponding to the eigenvalue  $\Lambda_3 = 1$  we easily obtain:  $N_3 = e_3$ .

The same procedure applied to yields the following unit eigenvectors of  $b$  (see, e.g. [1]):

$$n_1 = \frac{1}{\sqrt{1 + \Lambda_2}} \cdot e_1 + \frac{1}{\sqrt{1 + \Lambda_1}} \cdot e_2; \quad (7)$$

$$n_2 = \frac{1}{\sqrt{1 + \Lambda_1}} \cdot e_1 - \frac{1}{\sqrt{1 + \Lambda_2}} \cdot e_2;$$

$$n_3 = e_3.$$

The inclination angles  $\alpha$  and  $\beta$  of the vectors  $N_1$  and  $n_1$ , respectively, to  $x$ -axis result from

$$\tan \alpha = \sqrt{\Lambda_1} = \frac{\sqrt{4 + \gamma^2} + \gamma}{2}; \quad (8)$$

$$\tan \beta = \sqrt{\Lambda_2} = \frac{\sqrt{4 + \gamma^2} - \gamma}{2}.$$

so that (cf. [3])

$$\tan \alpha \cdot \tan \beta = 1. \quad (9)$$

In Fig. 2 they are plotted versus the shear angle. Thus, the eigenvectors of  $C$  and  $b$  rotate counterclockwise and clockwise, respectively, and consequently against each other. This rotation is illustrated in Figs. 3 and 4, where the eigenvectors of  $C$  and  $b$  are plotted for various values of the shear angle versus the corresponding deformed configuration of the rectangular sheet. One can also observe that the eigenvectors of  $b$  rotate in the same direction as the edges of the sheet while the eigenvectors of  $C$  in the opposite direction.

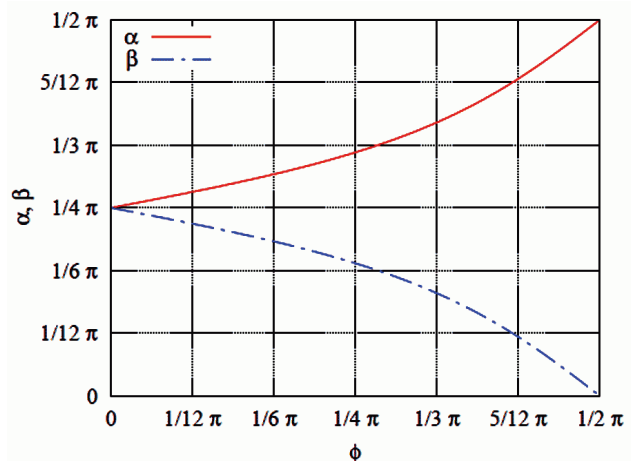


Figure 2 – Inclination angles  $\alpha$  and  $\beta$  of the first eigenvectors of  $C$  and  $b$ , respectively, versus the shear angle  $\phi$

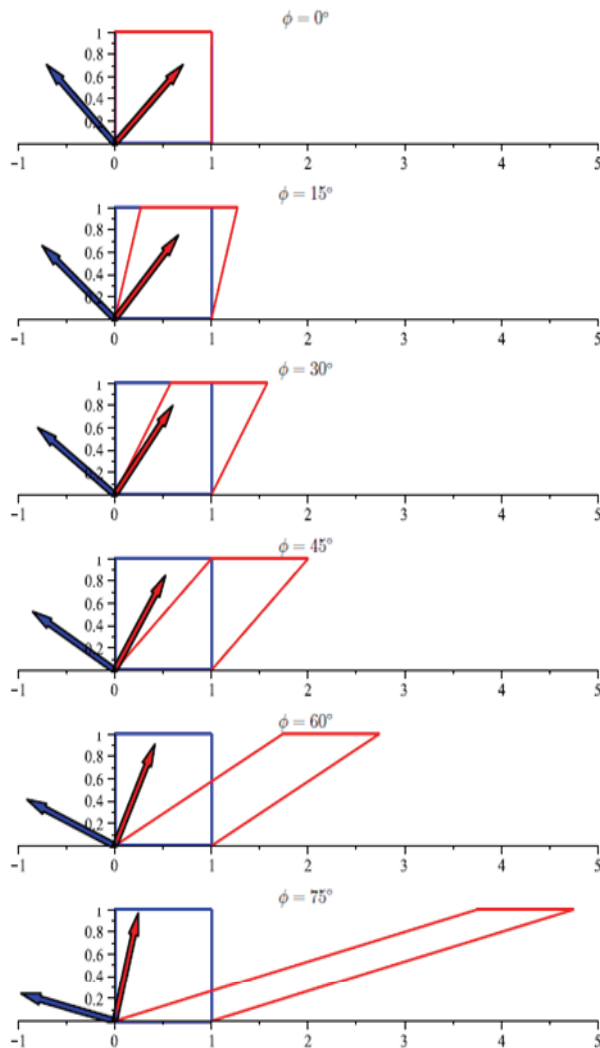


Figure 3 – Positions of the eigenvectors of  $\mathbf{C}$  for various values of the shear angle versus the deformed configuration of the rectangular sheet. The eigenvectors rotate counterclockwise

**Conclusion.** We studied rotations of principal strain directions under the kinematically defined simple shear. Accordingly, eigenvectors of the right and left Cauchy-Green tensors rotate against each other with the increasing amount of shear. Interestingly, the eigenvectors of  $\mathbf{b}$  rotate in the same direction as line elements of the material while the eigenvectors of  $\mathbf{C}$  in the opposite direction. For example, this can be important for anisotropic and in particular fiber reinforced materials. In this case, the direction of the maximal stretch will rotate with respect to reinforcement directions.

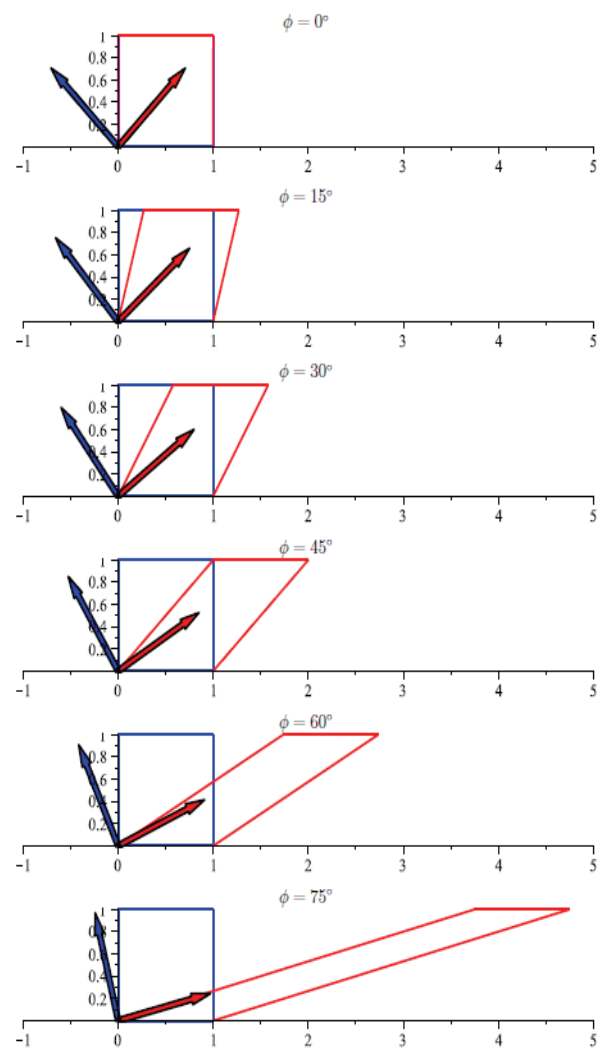


Figure 4 – Positions of the eigenvectors of  $\mathbf{b}$  for various values of the shear angle versus the deformed configuration of the rectangular sheet. The eigenvectors rotate clockwise

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