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An Approximate Analytical Solution of Vibration Problem for Imperfect FGM Shallow Shells with Time Dependent Thickness under Static Loading

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Abstract
This paper deals with research of nonlinear vibration of imperfect shallow shells made of functionally-graded materials (FGM) under static and dynamic loadings. The material properties are changing in the thickness direction according to the given power law distribution and the non-linear strain-displacement relationships based on the von Karman theory for moderately large normal deflections. Initial nonlinear system of differential equations transforms to singular ordinary differential equations with variable in time coefficients, which is solved by hybrid perturbation and WKB- Galerkin methods in three steps. Comparison of numerical integration of initial equation and a3proximate analytical solutions are given.
Keywords
Asymptotic approach, nonlinear dynamic problem, FGM shallow shells, time dependent parameters

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Introduction

Thin walled structures made of functionally graded materials (FGM) with metal inner surface and ceramic in outer surface are widely used, for example, in modern air-space systems, shipbuilding, electronics and other fields of science and technology because of their flexibility in design to have desired strength and durability.

FGMs are the heterogeneous composite materials in which the material properties are changing in the thickness direction or discontinuous as a stepwise gradation of the material constituents [1,2,4,11]. Composite shell structures which are used for modern flying apparatus undergo large deflection and/or static and nonlinear dynamic external mechanical loads. That's why it is important to take into account the geometrically nonlinear effects to ensure more accurate structural analysis and design. In recent years important studies about vibration and stability of FGM plates and shells under static, dynamic loading and in high temperature environment have been carried out, with using mostly numerical approaches [1-6] and just few ones deal with asymptotic approaches [12-15], because of the difficulties in numerical calculations.

The present work deals with an approximate analytical solution of nonlinear dynamic problem of FGM imperfect shallow shells based upon the von Karman theory for moderately large normal deflections with time dependent parameters (for example, with thickness depending on time) on the basis of hybrid (P-WKB-G) asymptotic method, which was successfully applied earlier [7-10]

1. Formulation of the problem. An approximate analytical solution

Suppose that the FGM imperfect shallow shell is simply supported at its edges and subjected to a transverse load $q_0(t)$ and compressive edge loads $r_0(t)$, $p_0(t)$. We assume that modulus of elasticity and the mass density changes in the thickness direction, while the Poisson ratio is assumed to be constant and thickness of shell is function of time.

Consider that initial imperfections in the middle shell surface, a system of nonlinear differential equations for functions of the normal stress and displacement based on the theory von Karman for large deflection and small strain are:

$$\rho_1 \frac{\partial^2 w}{\partial t^2} + \frac{E_1E_3 - E_2^2}{E_1(1-\nu^2)}\Delta\Delta(w-w_0) + 2\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\frac{\partial^2 w}{\partial x_1 \partial x_2} - \frac{\partial^2 \phi}{\partial x_2^2}\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 \phi}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} - k_2\frac{\partial^2 \phi}{\partial x_1^2} - k_1\frac{\partial^2 \phi}{\partial x_2^2} = q_0$$

$$\frac{1}{E_1}\Delta\Delta\phi = -k_1\frac{\partial^2 (w-w_0)}{\partial x_2^2} - k_2\frac{\partial^2 (w-w_0)}{\partial x_1^2} + \left[\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 w}{\partial x_1^2}\cdot\frac{\partial^2 w}{\partial x_2^2}\right] - \left[\left(\frac{\partial^2 w_0}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 w_0}{\partial x_1^2}\cdot\frac{\partial^2 w_0}{\partial x_2^2}\right] = 0, \quad (1)$$

where q_0 is intensity of transverse load, φ is stress function w is deflection.

The deflection function $w = (x_1, x_2, t)$ is chosen here as

$$w(x_1, x_2, t) = f(t)\sin\frac{m\pi x_1}{a}\sin\frac{n\pi x_2}{b} \quad (2)$$

The regular initial imperfection of middle surface of shell are taken in the form

$$w_0(x_1, x_2) = f_0 \sin\frac{m\pi x_1}{a}\sin\frac{n\pi x_2}{b} \quad (3)$$

where f_0 is the amplitude of initial imperfection.

Then, applying the Bubnov- Galerkin procedure to the Karman equations (1), we obtain the non-linear second order ordinary differential equation with variable in time coefficients for function $f(t)$ in the following form [1]:

$$\varepsilon^2\frac{d^2 f}{dt^2} + f\left(1 + 2f_0\bar{A}_2(t) - \bar{A}_3(t)f_0^2 - \bar{A}_1(t)\right) + f^2\left(-3\bar{A}_2(t)\right) + f^3\bar{A}_3(t) = Q_0 - \bar{A}_0(t) + f_0 - \bar{A}_2(t)f_0^2 \quad (4)$$

where

$$\varepsilon^2 = \frac{1}{\omega_{mn}^2},\ A_0(t) = \frac{16h(t)}{\pi^2 mn}(k_1 r_0 + k_2 p_0),\ A_1(t) = \frac{\pi^2 h(t)}{a^2}\left(m^2 r_0 + n^2\lambda^2 p_0\right)$$

$$A_2(t) = \frac{16E_1(t)mn\lambda^2\left(k_1 n^2\lambda^2 + k_2 m^2\right)}{3a^2\left(m^2 + n^2\lambda^2\right)^2},\ A_3(t) = \frac{512E_1(t)m^2n^2\lambda^4}{9a^4\left(m^2 + n^2\lambda^2\right)^2},\ \bar{A}_i = \frac{A_i}{\omega_{mn}^2} \quad (5)$$

$$\omega_{mn}^2 = \frac{1}{\rho_1(t)}\left[\frac{\left(E_1E_3 - E_2^2\right)}{E_1\left(1-\nu^2\right)}\cdot\frac{\left(m^2 + n^2\lambda^2\right)\pi^2}{a^4} + \frac{E_1\left(k_1 n^2\lambda^2 + k_2 m^2\right)^2}{\left(m^2 + n^2\lambda^2\right)^2}\right]$$

$$E_1(t) = \left(E_m + \frac{E_c - E_m}{k+1}\right)h(t),\ \rho_1 = \left(\rho_m + \frac{\rho_c - \rho_m}{k+1}\right)h(t),$$

k_1, k_2 are curvatures of middle surface shell in x_1 and x_2 directions.

Basic differential equation (4) is rewritten in the form

$$\varepsilon^2\frac{d^2 f}{dt^2} + B_1(t)f + \mu\left(B_2(t)f^2 + B_3(t)f^3\right) = \bar{Q}_0(t), \quad (6)$$

where

$$B_1(t) = 1 + 2f_0\bar{A}_2(t) - \bar{A}_3(t)f_0^2 - \bar{A}_1(t),\ B_2(t) = \frac{-3}{\mu}\bar{A}_2(t), B_3(t) = \frac{1}{\mu}\bar{A}_3(t) \tag{7}$$
$$\bar{Q}_0(t) = Q_0 - \bar{A}_0(t) + f_0 - \bar{A}_2(t)f_0^2$$

Here ε, μ are small parameters.

According to the perturbation method with respect to the parameter of nonlinearity μ, a solution of the differential equation (6) is presented in the form of the following two terms approximation [7]:

$$f(t) = \varphi_0(t) + \mu\varphi_1(t) \tag{8}$$

Substituting (7) into equation (5) and acquainted the terms with the same order of the small parameter we obtain the system equations for unknown functions $\varphi_0(t)$ and $\varphi_1(t)$:

$$\mu^0 : \varepsilon^2\varphi_0''(t) + B_1(t)\varphi_0 = \bar{Q}_0 \tag{9}$$
$$\mu^1 : \varepsilon^2\varphi_1''(t) + B_1(t)\varphi_1 = -B_2(t)\varphi_0^2 - B_3(t)\varphi_0^3 \tag{10}$$

The system of ordinary singular differential equations with variable in time coefficient B_1 is solved by two terms WKB-approximation [5]. Finally we have obtained the solution of nonlinear problem on the basis of the perturbation -two-terms WKB method as

$$\begin{aligned} f(t) &= \varphi_0(t) + \mu\varphi_1(t) = \sin K(t)\left(c_1 + \bar{c}_1(t)\right) + \cos K(t)\left(c_2 + \bar{c}_2(t)\right) + \\ &+ \mu\left(\sin K(t)\left(d_1 + \bar{d}_1(t)\right) + \cos K(t)\left(d_2 + \bar{d}_2(t)\right)\right) = \\ &= B_1(t)^{0.25}\left\{sinK(t)\left[c_1 + \bar{c}_1(t) + \mu\bar{d}_1(t)\right] + cosK(t)\left[c_2 + \bar{c}_2(t) + \mu\bar{d}_2(t)\right]\right\}. \end{aligned} \tag{11}$$

where

$$K(t) = \int \varepsilon^{-1}B_1^{\frac{1}{2}}(t)dt \tag{12}$$
$$\bar{c}_1(t) = \varepsilon\int\frac{\bar{Q}_0(t)\cos K(t)}{B_1^{-0.25}(t)}dt,\quad \bar{c}_2(t) = -\varepsilon\int\frac{\bar{Q}_0(t)\sin K(t)}{B_1^{-0.25}(t)}dt \tag{13}$$
$$\bar{d}_1(t) = \varepsilon\int\frac{\left(-B_2(t)\varphi_0^2 - B_3(t)\varphi_0^3\right)\bar{Q}_0(t)\cos K(t)}{B(t)^{-0.25}}dt \tag{14}$$
$$\bar{d}_2(t) = -\varepsilon\int\frac{\left(-B_2(t)\varphi_0^2 - B_3(t)\varphi_0^3\right)\bar{Q}_0(t)\sin K(t)}{B_1(t)^{-0.25}}dt \tag{15}$$

Initial conditions are taken in the following form:

$$\begin{aligned} \varphi(0) &= 1, \\ \varphi'(0) &= 0. \end{aligned} \tag{16}$$

In the third step of the hybrid (P-WKB-G) asymptotic method we will keep the perturbation functions but replaced the gauge functions by new amplitudes which depend on ε. In the Bubnov-Galerkin orthogonally principle one seeks an approximate solution in the form of a linear combination of specified (known) coordinate functions with unknown amplitude δ_0 which is a function of ε :

$$f_H(t,\varepsilon) = \exp\int \delta_0(\varepsilon)\varphi_0(t)dt, \tag{17}$$

where

$$\delta_{01,2} = -\frac{Q_0(b) - Q_0(a)}{\pm 4\int_a^b \left(iQ_0(t)^{\frac{3}{2}} \right) dx} \pm \sqrt{\left(\frac{Q_0(b) - Q_0(a)}{4\int_a^b \pm \left(iQ_0(t)^{\frac{3}{2}} \right) dx} \right)^2 - \frac{1}{\varepsilon^2}} \tag{18}$$

The result of three-step hybrid asymptotic solution of initial non- homogeneous nonlinear differential equation with variable coefficients for function $f^H(t)$ is given in the following form:

$$\begin{aligned} f^H(t) = B_1(t)\{ & \sin I^H(t, \delta_{0_1})[c_1 + \bar{c}_1(Q_0(t), N(f_0))] + \\ & + \cos I^H(t, \delta_{0_2})[c_2 + \bar{c}_2(Q_0(t), N(f_0))]\} \end{aligned}, \tag{19}$$

where

$$I^H(t) = \pm\int \delta_{0_{1,2}} iQ_0(t)^{1/2} dt. \tag{20}$$

2. Influence of static loading and imperfections of the middle shell surface

In this section we presented the influence of static loading and imperfections of the middle shell surface, where thickness is given in form (21). In Figure 1 there is comparison of the obtained asymptotic solution with direct numerical calculation of initial equation, where parameter of static loading is equal to 0.5, amplitude of imperfection is equal to 0.1. In Figure 2 there is a mold of imperfect shell under static loading, where forced vibration function and parameter a are given by relations (22), (23) respectively.

$$h(t) = h_0(1 - \eta t) \tag{21}$$

$$Q_0 = Sin\Omega t \tag{22}$$

$$a = k_1 p_0 + k_2 r_0 \tag{23}$$

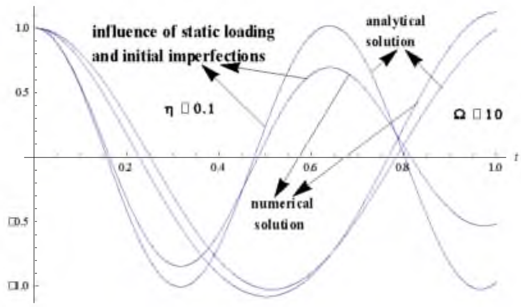

Figure 1. Influence of static loading and initial imperfection parameters

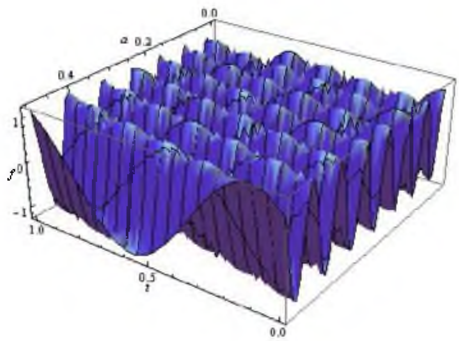

Figure 2. Mold of imperfect shell under static loading

Conclusions

An approximate analytical solution for forced vibrations of non-linear FGM shallow cylindrical shells with time dependent thickness on the basis of hybrid perturbation-two-terms WKB approximation method are obtained. A hybrid asymptotic technique leads to effective results both for “small”, and for “large” values of singular parameter at higher derivative of initial equation. For some particular parameters of structure analytical solutions are in a good enough correlations with direct numerical simulation of initial nonlinear differential equations with variable in time coefficients. Further investigations will be devoted to analyze the nonlinear dynamic behavior of complex shape shell made of FGM under external temperature loading.

References

[1] Dao HuyBich; Yu Do Long: Nonlinear Dynamical Analysis of Imperfect Functionally Graded Materials Shallow Shells. *Vietnam Journal of Mechanics*, *VAST*, 2010, p. 1-14.

[2] Vu Thi Thuy Anh; Nguyen Dinh Duc: Nonlinear Axisymmetric Response of Thin FGM Shallow Spherical Shells with Ceramic-Metal-Ceramic Layers under External Pressure and Temperature. *VNU Journal of Mathematics – Physics*, 2013, p. 1-15.

[3] Katarzhyna Kowal-Michalska; Radoslaw J. Mania: Static and Dynamic Thermomechanical Buckling Load of Functionally Graded Plate. *Mechanics and Mechanical Engineering*, 2013, Vol.112, p. 99-112.

[4] Khaire, P.D.; Ambhore, N.H.; Jagtap, K.R.: Nonlinear Free Vibration Response of FGM Cylindrical Shell in Thermal Environment. *Int. Journal on Mech. Engineering and Robotics*, 2014, Vol. 2.

[5] Kurpa, L.V.;Osetrov, A.A.; Shmatko, T.V.: Determination of eigen frequencies of functionally graded material shallow shells with using of R-function and spline-approximation. *Bulletin of NTU KhPI*, 2014, Vol. 6, p. 99-111.

[6] Stronzzi M, Pellicano F.:Nonlinear Vibrations of Functionally Graded Shells Subjected Harmonic External Load. *Proc. of the 4-th Int. Conf on Nonlinear Dynamics, Sevastopol*, 2014, p.341-346.

[7] Geer, J.F.; Andersen, C.V. Natural Frequency Calculations Using a Hybrid Perturbation Galerkin Technique. *Pan-American Congress on Appl. Mech.*, 1991, p. 571-574.

[8] Gristchak,V.Z.; Ganilova,O.A.: A Hybrid WKB-Galerkin Method Applied to a Piezoelectric Sandwich Plate Vibration Problem Considering Shear Force Effects. *Journal of Sound and Vibration*, 2008, Vol. 317, p. 366-377.

[9] Gristchak, V.Z.; Fatieieva, Ju.A.: An approximate nonlinear dynamic problem solution of functionally graded material shallow shell structure with in time thickness variation. *ZNU*, 2014, Vol. 3.

[10] Gristchak, V.Z.; Fatieieva, Ju.A.: An approximate analytical solution for nonlinear FGM Shell structure with variable in time parameters (To be published Proc. ECCOMAS Congress 2016, VII European Congress on Computational Methods in Applied Sciences and Engineering, Crete Island, Greece, 5–10 June 2016).

[11] K. Swaminathan, D.T. Naveenkumar, Stress, vibration and buckling analyses of FGM plates—A state-of-the-art review. *Composite Structures,* 2015, Vol. 120, p. 10–31.

[12] Obata Y, Noda N. Transient thermal stresses in a plate of functionally gradient material. *Ceramic Trans Funct Grad Mater,* 1993, Vol. 34, p.403–10.

[13] Aboudi J, Arnold SM, Pindara MJ. Response of functionally graded to thermal gradients. *Compos. Eng.*, 1994, Vol. 4(1), p.1–18.

[14] Aboudi J, Pindera MJ, Arnold SM. Thermo-inelastic response of functionally graded composites. *Int J Solids Struct.,* 1995, Vol. 32(12), p. 675–710.

[15] Aboudi J, Pindera MJ, Arnold SM. A coupled higher-order theory for functionally graded composites with partial homogenization. *Compos. Eng.,* 1995, Vol. 5(7), p. 771–92.