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FINITE-STEP METHOD FOR DETERMINING EQUILIBRIUM STATE OF GYROTHEODOLITE

The problem of orientation of a solid by using a torsion suspended gyrotheodolite is considered. Such gyrotheodolites are widely used in modern technology. During their operation, the problem arises of identifying the equilibrium position. It can be solved in many ways. A method is proposed for identifying the equilibrium position of a gyrotheodolite, which has several advantages over other well-known classical methods (least squares method, Kalman filter, and others). A mathematical description of the gyrotheodolite rotor motion is provided, a mathematical model of the method is given, and further development of the research is indicated.

Key words: gyrotheodolite, azimuth, gyroscope, inertial moment, damping moment, directing moment, moment from other unaccounted process forces.

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КІНЦЕВО-КРОКОВИЙ МЕТОД ВИЗНАЧЕННЯ РІВНОВАЖНОГО ПОЛОЖЕННЯ ГІРОТЕОДОЛІТА

Розглядається задача орієнтації твердого тіла за допомогою гіротеодоліта на торсіонному підвісі. Такі гіротеодоліта мають широке застосування в сучасній техніці. При їх роботі виникає задача ідентифікації положення рівноваги. Вона може вирішуватися багатьма способами. Запропоновано метод ідентифікації рівноважного положення гіротеодоліта, який має ряд переваг перед іншими відомими класичними методами (методом найменших квадратів, фільтром Калмана та іншими). Викладено математичний опис руху ротора гіротеодоліта, дана математична модель методу і позначено подальший розвиток даних досліджень.

Ключові слова: гіротеодоліт, азимут, гіроскоп, інерційний момент, демпфуючий момент, спрямовуючий момент, момент від інших неврахованих сил процесу.

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КОНЕЧНО-ШАГОВЫЙ МЕТОД ОПРЕДЕЛЕНИЯ РАВНОВЕСНОГО ПОЛОЖЕНИЯ ГИРОТЕОДОЛИТА

Рассматривается задача ориентации твёрдого тела с помощью гиротеодолита на торсионном подвесе. Такие гиротеодолиты имеют широкое применение в современной технике. При их работе возникает задача идентификации положения равновесия. Она может решаться многими способами. Предложен метод идентификации равновесного положения гиротеодолита, который имеет ряд преимуществ перед другими известными классическими методами (методом наименьших квадратов, фильтром Калмана и другими). Изложено математическое описание движения ротора гиротеодолита, дана математическая модель метода и обозначено дальнейшее развитие данных исследований.

Ключевые слова: гиротеодолит, азимут, гироскоп, инерциальный момент, демпфирующий момент, направляющий момент, момент от прочих неучтённых сил процесса.

Introduction. Nowadays the majority of the problems of identification of the parameters of orientation systems by the measurement results are solved using the algorithms based on various numerical methods. When developing a competitive numerical method one needs to provide for a series of requirements which are due to the features of the specific practical problem considered. When identifying *the gyrotheodolite equilibrium position* it is of particular importance to meet the following requirements:

– providing the identification accuracy;
– the minimal amount of measurements;
– the algorithm performance;
– stability of the computational process;
– the algorithm simplicity;
– low interference susceptibility.

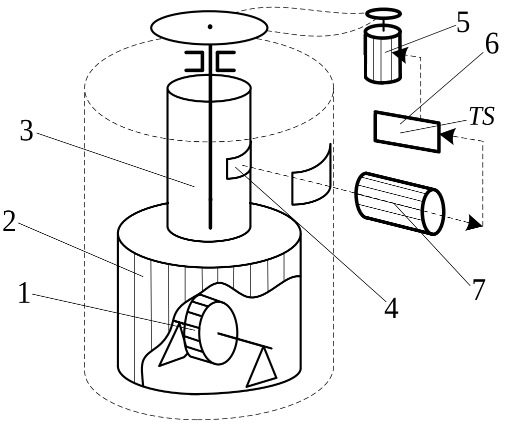

Fig. 1 – Gyrotheodolite model.

Previous to developing a mathematical model of the method proposed in this paper we consider the features of the mathematical description of the gyrotheodolite rotor motion.

A gyrotheodolite is commonly used for determining the *azimuths* of directions on the Earth’s surface. By its construction a gyrotheodolite is an *angle gauge* incorporating a *gyroscope*, which is the part of the device responsible for determining the direction of the true meridian, and a theodolite [1].

The problem of determining the azimuth arises in various applications, such as, for instance, navigation of aircrafts, ships, submarines, during surveying, mine tunneling, etc. For solving this problem gyroscopic azimuth-orientation instruments are widely used, which have several advantages over the astronomical, magnetic and other methods, such as the capability to operate in any season and at any time of the day, inside closed objects including indoor spaces [2].

In the majority of the gyrotheodolite models available the sensitive element, which is directly involved in determining the true meridian plane, consists of a torsion suspended pendulous gyroscope with automated tracking system (fig. 1).

Gyrorotor 1 in casing 2 is suspended by a flexible tape called torsion 3. In order to reduce the uncertainty torque caused by the torsion elastic torque (*torsion bar spring tension*) a *tracking system* or torsion support system is introduced, which rotates the torsion upper end fixation unit following the motion of the sensitive element. The tracking system is composed of mirror 4, error sensor 7, amplifier 6 and drive 5 connected to torsion upper end fixation unit 3 [2].

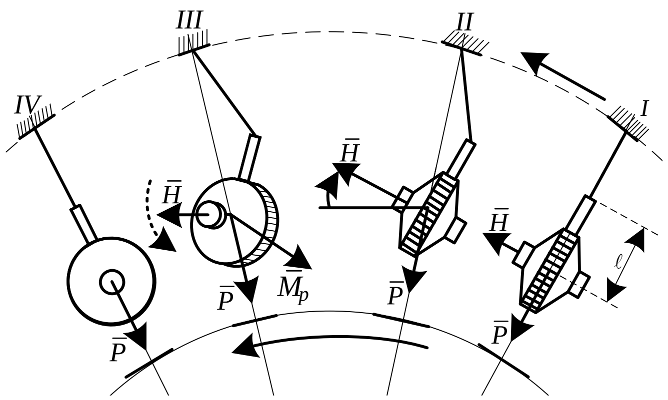

Fig. 2 – Gyrotheodolite operation principle.

To simplify the explanation of the operation principle of a torsion suspended two-degree-of-freedom gyrotheodolite pendulum let us assume that the gyrotheodolite is placed at the Equator (fig. 2) and the main gyroscope axis is horizontal and aligned with the West-East direction at the initial moment of time, besides the kinetic momentum is directed eastwards (position I, fig. 2). In this case the sensitive element gravity center is in the vertical plane and, hence, does not generate a moment about the gyromotor axis. Due to the daily rotation of the Earth the horizon plane as well as the direction of the local vertical changes its location with respect to the inertial space continuously. At the same time the direction of the gyrorotor axis stays unchanged in the inertial space, since no momentum is applied to the gyroscope in position I, fig. 2. Hence, the main axis deviates from the local horizon plane (position II, fig. 2), besides the gyroscope rotates about the point at which it is attached to the torsion. The gravitational moment generated thereby $M_p = P\ell\beta$, where β is the angle of deviation of the gyroscope main axis from the meridian plane (position II, fig. 2), induces the gyroscope precession in the direction which takes the kinetic moment $\bar{H}$ to get aligned with the meridian plane (position IV, fig. 2). In order for the gyroscope main axis to keep its northwards direction after being aligned with the meridian plane it needs to rotate with the same angular velocity as the meridian plane, i.e. $\omega_E \sin\varphi$ at the latitude φ, where ω_E is the angular velocity of the Earth.

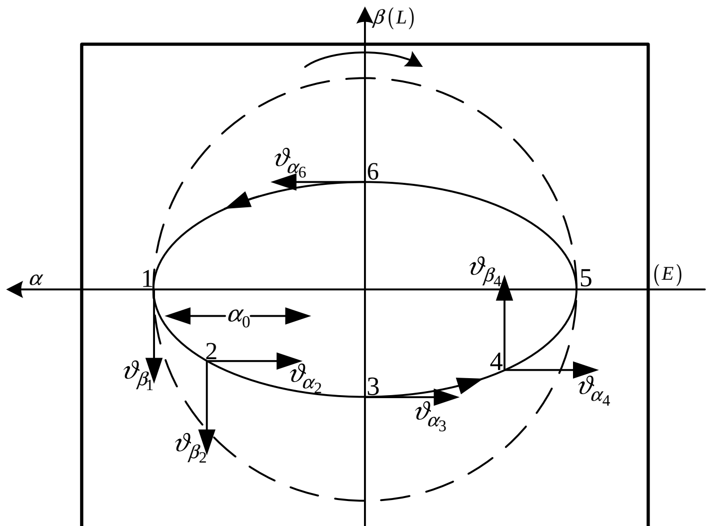

Fig. 3 – Gyrotheodolite main axis precession scheme.

Thus at the deviation by angle α_0 (or β_0) the gyro compass executes undamped oscillations about its equilibrium position, which lies in the meridian plane, moreover the trajectory of the gyroscope axis is an ellipse which semi-major axis is in the horizon plane and semi-minor axis is in the vertical plane. As a rule the ratio $\alpha_{\max} / \beta_{\max}$ reaches $200-500$ and the oscillation period is tens of minutes. Since the vertical component of the angular velocity of the Earth at the Equator is zero: $\varphi = 0^\circ$, $\omega_\beta = \omega_E \sin\varphi = 0$, the middle position of the oscillations of the gyroscope in the β angle (fig. 3) lies in the horizon plane. At arbitrary latitude the trajectory (fig. 4) of the gyroscope axis is also an ellipse with center shifted up by β^* (for northern latitude).

Thus the gravitational moment needs to give rise to gyroscope precession in azimuth with the velocity $\omega_{pr} = \omega_E \sin\varphi$ relative to the inertial space:

$$\omega_{pr} = \frac{M_p}{H} \quad \text{or} \quad \omega_E \sin\varphi = \frac{P\ell\beta^*}{H}, \tag{1}$$

$$\text{hence,} \quad \beta^* = \frac{H}{P\ell}\omega_E \sin\varphi . \qquad (2)$$

At the deviation of the main axis from the horizon plane the angle velocity of the gyroscope precession is the same as the velocity of the meridian plane rotation. Nevertheless, there occur the oscillations of the gyroscope main axis relative to the meridian plane, which origins are considered below.

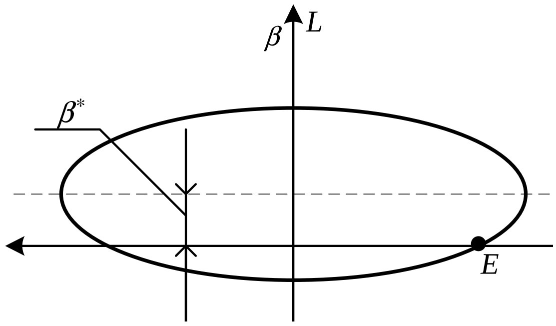

Fig. 4 – Trajectory of gyroscope axis at arbitrary latitude.

Assume that at the initial moment of time the gyroscope main axis is in the horizon plane at the Equator and it is deviated from the meridian plane by the angle α_0 (point 1 in fig. 3). Since at this position the angle $\beta = 0$ and the pendulosity $M_p = 0$, the gyroscope acts as a free gyroscope, i.e. the positive end of its main axis falls relative to the horizon plane due to the Earth's rotation. The deviation of the gyroscope main axis by the angle β from the horizon plane results in the gyroscope precession in the direction of alignment of the main axis with the meridian plane due to the pendulosity $M_p = P\ell\beta$ (point 3 in fig. 3).

In the position corresponding to point 3 (fig. 3) the angle β of deviation of the gyroscope from the horizon plane reaches its maximum modulo and the velocity of the gyroscope is zero, hence, it passes this position.

On segment 3 – 5 (fig. 3) the horizon plane rotates towards the gyroscope main axis. Thus the angle β decreases which results in the decreasing pendulosity M_p; the eastward precession of the gyroscope main axis slows down and at point 5 (fig. 3) the precession angle velocity becomes zero, the gyroscope deviation from the meridian plane reaches its maximum. Further on the rotation of the horizon plane causes the rise of the gyroscope main axis above the horizon plane, the angle β increases thus bringing about the gyroscope precession. The pendulosity is now of the opposite sign which again results in gyroscope precession relative to meridian plane (segment 5 – 6, fig. 3). At point 6 (fig. 3) the deviation of the gyroscope main axis form the horizon plane reaches its maximum one more time, thus the gyroscope moves westwards with the maximal speed, hence, its main axis passes the meridian plane.

On segment 6 – 1 (fig. 3) the horizon plane moves towards the gyroscope main axis, hence, the angle β and the pendulosity M_p gradually decrease, the gyroscope precession slows down and its angular velocity becomes zero at point 1 (fig. 3) [3].

In the absence of the momenta other than pendulosity the trajectory of the gyroscope axis is an ellipse. In reality, since the presence of the momenta (such as suspension friction moment, torsion bar spring tension moment, etc.) is inevitable, the oscillations of the gyroscope axis decay in time, nevertheless the decay process lasts for quite a long time. That is why the computational algorithms for quick identification of the true meridian are required, which allow for solving the problem with the given accuracy and in short time at the maximal volume of measurements of the angle α.

The motion of the gyrotheodolite main axis is approximately described by the following equation [4]:

$$A\ddot{\alpha} + D\dot{\alpha} + HU\cos\varphi\sin\varphi = M , \qquad (3)$$

where $A\ddot{\alpha}$ is the inertia moment; $D\dot{\alpha}$ is the damping moment; $HU\cos\varphi\sin\varphi$ is the meridian alignment moment; M stands for the moment resulting from other unaccounted forces; U is the gyrotheodolite velocity.

Equation (3) can be considered as a particular case of *an automatic control system (ACS).*

The problem of identification of system parameters by the output signal measurement results is one of the relevant identification problems, which in the case of equation (1) consists in identifying the true meridian or the angle α equilibrium.

In those cases when in the process of problem solving the volume of the stored information and processing time are strictly regulated applying the classical methods (such as, for instance, Kalman filter, etc.) can't be justified. Then the finite-step method proposed in this paper can be successfully applied, which provides a solution after a finite predetermined number of computations without any iterative procedure involved. At the same time the solution obtained by this method is mathematically exact.

We need to mention here that the algorithm proposed provides a solution to the identification problem for a real ACS with the given accuracy in the case when its mathematical model slightly differs from the real one.

Problem setting. Main and secondary problems. The solution to equation (1) and those which are similar to it can be given by the sum of a constant R and *decaying* sinusoids as follows:

$$\alpha(t) = R + \sum_{k=1}^{N} A'_k R^{\beta_k t} \sin(\omega_k t + \psi_k), \quad (4)$$

where β_k can be either positive or zero [4].

Since in the case of gyrotheodolite and some ACS's the constant is the main object to be identified, the problem of computing R with high accuracy in limited time is referred to as *the main problem* below. Then *the secondary problem* consists in identifying the parameters A'_k, β_k, ω_k, ψ_k. The algorithm presented in the paper provides for solving both of these problems. Nevertheless, the main focus is on the substantiation of the main problem with outlining the ways for solving the secondary one. Let us name the method presented *the method of equidistant points*.

Mathematical model. Method of equidistant points. Solving the main problem, i.e. the problem of identifying the constant R in (4), requires sufficient number N_1 of measurements $\alpha(t_j)$. The number N_1 depends on the quantity of the unknown parameters in (4) which need to be identified. We assume that the measurements are taken at regular intervals of time, then

$$t_j = t_0 + j\Delta t, \quad (j = 0, 1, 2, ..., n), \quad (5)$$

where t_0 and t_j denote the moments of the initial and current measurements; Δt is the given step of measurements; j is the measurement number.

Let us choose a conditional midpoint on the measurement time interval from t_0 to t_N:

$$t_c = \frac{t_0 + t_p}{2} = t_0 + p'\Delta t, \quad (6)$$

where $p = 2p' < N_1$. The specific values of p' and N_1 are determined below.

In what follows we operate with the values t_j and t_{p-j} symmetric with respect to the midpoint.

From (5) and (6) it follows that:

$$t_c = \frac{t_j + t_{p-j}}{2}; \quad t_{p-j} = 2t_c - t_j,$$

that is why the method of solving the main problem given below is called the method of equidistant points:

t_{p-j} and t_j are distanced equally from t_c.

Assuming that the measurement $\alpha(t)$ can be given analytically by (4) we write down the differences and sums of measurements for equidistant points:

$$\alpha_{p-j} - \alpha_j \equiv \alpha_{p'+(p'-j)} - \alpha_{p'-(p'-j)} = R + \sum_{k=1}^{N} A'_k e^{-\beta_k t_{p-j}} \sin(\omega_k t_{p-j} + \psi_k) - R - \sum_{k=1}^{N} A'_k e^{\beta_k t_j} \sin(\omega_k t_j + \psi_k) =$$

$$= \sum_{k=1}^{N} A'_k \left[e^{-\beta_k t_{p-j}} \sin(\omega_k t_{p-j} + \psi_k) - e^{-\beta_k t_j} \sin(\omega_k t_j + \psi_k) \right]; \quad (7)$$

$$\alpha_{p-j} + \alpha_j \equiv \alpha_{p'+(p'-j)} + \alpha_{p'-(p'-j)} = 2R + \sum_{k=1}^{n} A'_k \left[e^{-\beta_k t_{p-j}} \sin(\omega_k t_{p-j} + \psi_k) + e^{-\beta_k t_j} \sin(\omega_k t_j + \psi_k) \right]. \quad (8)$$

The dependence of t_j and t_{p-j} on the midpoint t_c is given by:

$$t_j = t_0 + j\Delta t = t_0 + p'\Delta t - (p' - j)\Delta t = t_c - (p' - j)\Delta t;$$

$$t_{p-j} = t_0 + (p - j)\Delta t = t_0 + p'\Delta t + (p' - j)\Delta t = t_c + (p' - j)\Delta t. \quad (9)$$

We transform the expression in square brackets in (7) using *the Euler formulae*:

$$e^{-\beta_k t_{p-j}} \sin(\omega_k t_{p-j} + \psi_k) - e^{-\beta_k t_j} \sin(\omega_k t_j + \psi_k) = \exp\left[-\beta_k t_c - \beta_k (p' - j)\Delta t\right] \sin\left[\omega_k t_c + \omega_k (p' - j)\Delta t + \psi_k\right] -$$

$$- \exp\left[-\beta_k t_c + \beta_k (p' - j)\Delta t\right] \sin\left[\omega_k t_c - \omega_k (p' - j)\Delta t + \psi_k\right] = \exp\left[-\beta_k t_c - \beta_k (p' - j)\Delta t\right] \times$$

$$\times \left\{ \exp\left[(\omega_k t_c + \omega_k (p' - j)\Delta t + \psi_k) \cdot i \right] - \exp\left[(\omega_k t_c + \omega_k (p' - j)\Delta t + \psi_k)(-i) \right] \right\} \frac{1}{2i} - \exp\left[-\beta_k t_c + \beta_k (p' - j)\Delta t\right].$$

$$\left\{ \exp\left[(\omega_k t_c - \omega_k (p' - j)\Delta t + \psi_k) i \right] - \exp\left[(\omega_k t_c - (p' - j)\Delta t + \psi_k)(-i) \right] \right\} \frac{1}{2i} =$$

$$= \exp\left[-\beta_k t_c + i(\omega_k t_c + \psi_k)\right] \cdot \sin\left[(p' - j)\Delta t(\omega_k + i\beta_k)\right] + \exp\left[-\beta_k t_c - i(\omega_k t_c - \psi_k)\right] \cdot \sin\left[(p' - j)\Delta t(\omega_k - i\beta_k)\right] =$$

$$= a'_k \sin\left[x'_k (p - j)\Delta t\right] + a''_k \sin\left[x''_k (p' - j)\Delta t\right], \quad (10)$$

where $a'_k = \exp\left[-\beta_k t_c + i(\omega_k t_c + \psi_k)\right]$; $a''_k = \exp\left[-\beta_k t_c - i(\omega_k t_c + \psi_k)\right]$; $x'_k = \omega_k + i\beta_k$; $x''_k = \omega_k - i\beta_k$; p' is the number of the measurement corresponding to the conditional midpoint.

Apparently the coefficients a'_k, a''_k, x'_k, x''_k are independent of the number j and, hence, are the same for any equidistant points α_{p-j} and α_j $(j = 1, 2, ...)$.

By analogy transform the expression in the square brackets in (8):

$$e^{-\beta_k t_{p-j}} \cdot \sin\left(\omega_k t_{p-j} + \psi_k\right) + e^{-\beta_k t_j} \cdot \sin\left(\omega_k t_j + \psi_k\right) = b'_k \cos\left[x'_k \left(p' - j\right)\Delta t\right] + b''_k \cos\left[x''_k \left(p' - j\right)\Delta t\right], \tag{11}$$

where $b'_k = -ia'_k$, $b''_k = ia''$.

In the case of zero coefficients $\beta_k = 0$ (10) and (11) are simplified to become:

$$\sin\left(\omega_k t_{p-j} + \psi_k\right) - \sin\left(\omega_k t_j + \psi_k\right) = 2\cos\left(\omega_k t_c + \psi_k\right)\sin\left(\omega_k \left(p' - j\right)\Delta t\right) = a'''_k \sin x'''_k \left(p' - j\right)\Delta t; \tag{12}$$

$$\sin\left(\omega_k t_{p-j} + \psi_k\right) + \sin\left(\omega_k t_j + \psi_k\right) = 2\sin\left(\omega_k t_c + \psi_k\right)\cos\omega_k \left(p' - j\right)\Delta t = b'''_k \cos x'''_k \left(p' - j\right)\Delta t, \tag{13}$$

where $a'''_k = 2\cos\left(\omega_k t_c + \psi_k\right)$, $b'''_k = 2\sin\left(\omega_k t_c + \psi_k\right)$, $\left(a''_k\right)^2 + \left(b'''_k\right)^2 = 4$, $x'''_k = \omega_k$.

Clearly, the coefficients a'''_k, b'''_k, x'''_k do not depend on the number in this case as well.

Comparing (10) and (12) we conclude that in the both cases $\beta_k \neq 0$ and $\beta_k = 0$ these expressions are of the same type with the only difference that for $\beta_k \neq 0$ we have two summands of the form $a_k \sin x_k \left(p' - j\right)\Delta t$ and for $\beta_k = 0$ only one such summand, where a_k and x_k are unknown values ($a_k = a'_k, a''_k$ or a'''_k; $x_k = x'_k, x''_k$ or x'''_k).

Similarly, (11) and (13) are of the same form $b_k \cos x_k \left(p' - j\right)\Delta t$ with the unknown b_k and x_k, moreover, the coefficients b_k can be written in terms of a_k. By the above argument, substituting (10) and (11) in (7) and (8) we get:

$$\delta_m \equiv \alpha_{p-j} - \alpha_j = \sum_{k=1}^{N} A_k \sin x_k \left(p' - j\right)\Delta t; \quad \eta_m \equiv \alpha_{p-j} + \alpha_j = 2R + \sum_{k=1}^{N} B_k \cos x_k \left(p' - j\right)\Delta t, \tag{14}$$

where $A_k = A'_k a_k$, $B_k = A'_k b_k = f\left(A_k\right)$.

Form (8) and (10) it follows that for each summand of the form $A'_k e^{-\beta_k t_c} \sin\left(\omega_k t_c + \psi_k\right)$ there are two corresponding summands of the form $A_k \sin x_k \left(p' - j\right)\Delta t$ in the first formula of (14) and two summands of the form $B_k \cos x_k \left(p' - j\right)\Delta t$ in the second formula of (14). Thus each decaying sinusoid is reduced to two complex sinusoids (cosinusoids) with complex amplitudes. From (12) and (13) we have that for each summand of the form $A_k \sin\left(\omega_k t_c + \psi_k\right)$ there is one corresponding summand in (14).

Hence, in (14) the number N depends on the expected measurement composition and equals:

$$N = 2n_1 + n_2, \tag{15}$$

where n_1 is the number of the decaying sinusoids, n_2 is the number of the sinusoids that do not decay in formula (4), which is given a priory.

Consequently, formulae (14) derived above contain $2N + 1$ unknowns: R, A_k, x_k, all of which are independent of the measurement number.

To simplify the further argument we introduce the following notations:

$$\Delta x_k = x_k \Delta t, \quad p' - i = m, \tag{16}$$

then the differences and sums of measurements are reduces to the form:

$$\delta_m \equiv \alpha_{p'+m} - \alpha_{p'-m} = \sum_{k=1}^{N} A_k \sin\left(\Delta x_k \cdot m\right); \quad \eta_m \equiv \alpha_{p'+m} - \alpha_{p'-m} = 2R + \sum_{k=1}^{N} B_k \cos\left(\Delta x_k \cdot m\right), \tag{17}$$

with the coefficients A_k, B_k introduced in (14).

In what follows we omit the prime symbol from the notations a_k and x_k keeping in mind that these coefficients are the same as given by (19) depending on the value of β_k.

Thus we arrive at:

$$\delta_m \equiv \alpha_{p'+m} - \alpha_{p'-m} \equiv \sum_{k=1}^{N} A_k \sin\left(\Delta x_k \cdot m\right); \tag{18}$$

$$\eta_m \equiv \alpha_{p'+m} + \alpha_{p'-m} \equiv 2R + \sum_{k=1}^{N} B_k \cos\left(\Delta x_k \cdot m\right), \tag{19}$$

where the unknowns R, A_k and Δx_k do not depend on the measurement number, B_k can be written in terms of A_k (see formulae (14)).

To identify the unknowns let us consequentially set $m \equiv 1, 2, 3, ..., N+1$ in (18) and write down the following system of equation:

$$\delta_m = \sum_{k=1}^{N} A_k \sin\left(\Delta x_k \cdot m\right) \quad (m \equiv 1, 2, 3, ..., N+1), \tag{20}$$

where A_k are the coefficients to be identified.

A non-trivial solution A_k exists if the following determinant equals zero [2]:

$$D_1 = \begin{vmatrix} \delta_{N+1} & \sin\left[\left(N+1\right)\Delta x_1\right] & \sin\left[\left(N+1\right)\Delta x_N\right] \\ \delta_N & \sin\left(N\Delta x_1\right) & \sin\left(N\Delta x_N\right) \\ \dots & \dots & \dots \\ \delta_2 & \sin\left(2\Delta x_1\right) & \sin\left(2\Delta x_N\right) \\ \delta_1 & \sin\Delta x_1 & \sin\Delta x_N \end{vmatrix} = 0\,, \tag{21}$$

which can be reduced to the form:

$$D_1^{(N)} = \begin{vmatrix} \delta_{N+1}^{(N)} & y_1^N & \dots\dots & y_N^N \\ \delta_N^{(N)} & y_1^{N-1} & \dots\dots & y_N^{N-1} \\ \vdots & \vdots & \dots\dots & \vdots \\ \delta_3^{(N)} & y_1^2 & \dots\dots & y_N^2 \\ \delta_2^{(N)} & y_1 & \dots\dots & y_N \\ \delta_1^{(N)} & 1 & \dots\dots & 1 \end{vmatrix} = 0\,, \tag{22}$$

where $y_k = \cos\Delta x_k$, and $\delta_m^{(N)}$ is given by the formula:

$$\delta_m^{(\ell)} = 2\delta_m^{(\ell-1)} \quad \left(m = 1, 2, ..., \ell;\ \ell = 2, 3, ..., N\right),$$

which is derived by transforming D_1 as shown below.

We apply the following transformations to the determinant D_1 given by (21) [5].

We first subtract its third row from the first one, the forth row from the second one, and so on, i.e. we subtract the $\left(n+2\right)-$th row of the determinant from its $n-$th row; the last row but one is kept unchanged and the last one is multiplied by 2.

Then in the $n-$th row of the determinant we get:

$$\sin\left(N+2-n\right)\Delta x_k - \sin\left(N+2-n-2\right)\Delta x_k = 2\cos\left(N-n+1\right)\Delta x_k \cdot \sin\Delta x_k\,.$$

We reduce each column by $2\sin\Delta x_k$ (assuming that $\Delta x_k \neq 0,\ k\pi$).

Thus,

$$D_1 = \begin{vmatrix} \delta_{N+1}^{(1)} & \cos N\Delta x_1 & \dots & \cos N\Delta x_N \\ \delta_N^{(1)} & \cos\left(N-1\right)\Delta x_1 & \dots & \cos\left(N-1\right)\Delta x_N \\ \dots & \dots & \dots & \dots \\ \delta_2^{(1)} & \cos\Delta x_1 & \dots & \cos\Delta x_N \\ \delta_1^{(1)} & 1 & \dots & 1 \end{vmatrix} = 0\,,$$

where $\delta_m^{(1)} = \delta_m - \delta_{m-2}$ $\left(m = 3, 4, ..., N+1\right)$, $\delta_2^{(1)} = \delta_2$, $\delta_1^{(1)} = 2\delta_1$.

Next we add successively the first row of the determinant with its third row, the second row with the forth row, and so on up to the sum of the $\left(N-1\right)-$st and $\left(N+1\right)-$st rows; whereby we get in the $n-$th row:

$$\cos\left(N+1-n\right)\Delta x_k + \cos\left(N+1-n-2\right)\Delta x_k = 2\cos\left(N-n\right)\Delta x_k \cos\Delta x_k\,.$$

Multiplying the $N-$th and $\left(N+1\right)-$st rows by 2 we then reduce each row of the determinant by 2:

$$D_1^{(2)} = \begin{vmatrix} \delta_{N+1}^{(2)} & y_1 \cos(N-1)\Delta x_1 & \ldots & y_N \cos(N-1)\Delta x_N \\ \delta_N^{(2)} & y_1 \cos(N-2)\Delta x_1 & \ldots & y_N \cos(N-2)\Delta x_N \\ \ldots & \ldots & \ldots & \ldots \\ \delta_3^{(2)} & y_1^2 & \ldots & y_N^2 \\ \delta_2^{(2)} & y_1 & \ldots & y_N \\ \delta_1^{(2)} & 1 & \ldots & 1 \end{vmatrix} = 0 ,$$

where $y_k = \cos \Delta x_k$ $(k = 1, 2, ..., N)$; $\delta_m^{(2)} = \delta_m^{(1)} + \delta_{m-2}^{(1)}$ $(m = 3, 4, ..., N+1)$; $\delta_2^{(2)} = 2\delta_2^{(1)}$; $\delta_1^{(2)} = 2\delta_1^{(1)}$.

Summing the determinant rows as above up to the sum of the $(N-2)-$nd and $N-$th rows and multiplying the rows with numbers N, $N-1$ and $N+1$ by 2, after reducing all the columns by 2 we arrive at the following equation:

$$D_1^{(3)} = \begin{vmatrix} \delta_{N+1}^{(3)} & y_1^2 \cos[(N-2)\Delta x_1] & \ldots & y_N^2 \cos[(N-2)\Delta x_N] \\ \delta_N^{(3)} & y_1^2 \cos[(N-3)\Delta x_1] & \ldots & y_N^2 \cos[(N-3)\Delta x_N] \\ \ldots & \ldots & \ldots & \ldots \\ \delta_4^{(3)} & y_1^3 & \ldots & y_N^3 \\ \delta_3^{(3)} & y_1^2 & \ldots & y_N^2 \\ \delta_2^{(3)} & y_1 & \ldots & y_N \\ \delta_1^{(3)} & 1 & \ldots & 1 \end{vmatrix} = 0 ,$$

where $\delta_m^{(3)} = \delta_m^{(2)} + \delta_{m-2}$ $(m = 4, 5, ..., N+1)$; $\delta_m^{(3)} = 2\delta_m^{(2)}$ $(m = 1, 2, 3)$.

By repeating the procedure we get:

$$D_1^{(N)} = \begin{vmatrix} \delta_{N+1}^{(N)} & y_1^N & \ldots & y_N^N \\ \delta_N^{(N)} & y_1^{(N-1)} & \ldots & y_N^{(N-1)} \\ \ldots & \ldots & \ldots & \ldots \\ \delta_3^{(N)} & y_1^2 & \ldots & y_N^2 \\ \delta_2^{(N)} & y_1 & \ldots & y_N \\ \delta_1^{(N)} & 1 & \ldots & 1 \end{vmatrix} = 0 , \quad (23)$$

where

$$\delta_m^{(\ell)} = \delta_m^{(\ell-1)} + \delta_{m-2}^{(\ell-1)} \quad (m = \ell+1, ..., N+1); \quad (24)$$

$$\delta_m^{(\ell)} = 2\delta^{(\ell-1)} \quad (m = 1, 2, ..., \ell;\ \ell = 2, 3, ..., N), \quad (25)$$

with $y_k = \cos \Delta x_k$, and $\delta_m^{(N)}$ given by formulae (24), (25).

Expanding determinant (23) by the elements of its first column and dividing the equation obtained by the cofactors of the element $\delta_{N+1}^{(N)}$ we have:

$$\sum_{i=1}^{N} z_i \delta_{N+1-i}^{(N)} = -\delta_{N+1}^{(N)}, \quad (26)$$

where

$$z_i = (-1)^i \cdot \frac{D_{i+1}}{D_1} \quad (i = 1, 2, 3, ..., N), \quad (27)$$

and the determinants D_{i+1} $(i = 0, 1, 2, 3, ..., N)$ are derived by deleting the $(i+1)-$st row from the table:

$$\begin{vmatrix} y_1^N & \ldots\ldots & y_N^N \\ y_1^{N-1} & \ldots\ldots & y_N^{N-1} \\ \ldots & \ldots & \ldots \\ y_1 & \ldots\ldots & y_N \\ 1 & \ldots\ldots & 1 \end{vmatrix}. \qquad (28)$$

The determinant D_1 obtained from table (28) by deleting its first row is in fact the Vandermonde determinant and can be computed by the formula [2]:

$$D_1 = (y_1 - y_2)(y_1 - y_3)\ldots(y_1 - y_N)(y_2 - y_3)\ldots(y_{N-1} - y_N). \qquad (29)$$

The determinant D_1 does not vanish if

$$y_k \neq y_m \quad \text{for} \quad k \neq m \quad (k, m = 1, 2, \ldots, N). \qquad (30)$$

Hence, if condition (30) holds then system (20) is reduces to a single linear equation (26) in N unknowns z_i.

By analogy one reduces the system of equations obtained from (19) for m equal to $0, 1, 2, \ldots, N$:

$$\eta_m = 2R + \sum_{k=1}^{N} B_k \cos(\Delta x_k \cdot m) \quad (m = 0, 1, 2, \ldots, N), \qquad (31)$$

where the coefficients B_k are unknown. A non-trivial solution B_k exists if the following determinant is zero:

$$D_2 = \begin{vmatrix} \eta_N - 2R & \cos(N\Delta x) & \ldots & \cos(N\Delta x_N) \\ \eta_{N-1} - 2R & \cos[(N-1)\Delta x_1] & \ldots & \cos[(N-1)\Delta x_N] \\ \ldots & \ldots & \ldots & \ldots \\ \eta_2 - 2R & \cos(2\Delta x_1) & \ldots & \cos(2\Delta x_N) \\ \eta_1 - 2R & \cos\Delta x_1 & \ldots & \cos\Delta x_N \\ \eta_0 - 2R & 1 & \ldots & 1 \end{vmatrix} = 0.$$

Arguing as above, the determinant D_2 can be reduced to the form:

$$D_2^{(N)} = \begin{vmatrix} \eta_N^{(N)} - 2^N R & y_1^N & \ldots & y_N^N \\ \eta_{N-1}^{(N)} - 2^N R & y_1^{N-1} & \ldots & y_N^{N-1} \\ \ldots & \ldots & \ldots & \ldots \\ \eta_1^{(N)} - 2^N R & y_1 & \ldots & y_N \\ \eta_0^{(N)} - 2^N R & 1 & \ldots & 1 \end{vmatrix} = 0, \qquad (32)$$

where

$$\eta_m^{(\ell)} = \eta_m^{(\ell-1)} + \eta_{m-2}^{(\ell-1)} \ (m = \ell, \ell+1, \ldots, N); \ \eta_m^{\ell} = 2\eta_m^{(\ell-1)} \ (m = 1, 2, \ldots, \ell; \ \ell = 2, 3, \ldots, N). \qquad (33)$$

We expand determinant (32) following the same procedure as for determinant (22) above to get:

$$\sum_{i=1}^{N} z_i \left(\eta_{N-i}^{(N)} - 2^N R\right) = -\left(\eta_N^{(N)} - 2^N R\right), \qquad (34)$$

where z_i are given by (27).

Introducing the notations $\varphi_i = \delta_{N+1-i}^{(N)}$; $Q = \eta_{N-i}^{(N)}$ for simplifying the representation, we reduce equations (26) and (34) to the form:

$$\sum_{i=1}^{N} z_i \varphi_i = -\varphi_0; \ \sum_{i=1}^{N} z_i \left(Q - 2^N R\right) = -\left(Q_0 - 2^N R\right). \qquad (35)$$

Thus constructing the linear combinations of differences (18) and sums (19) of measurements at equidistant points by formulae (25) and (33) we arrive at system of equations (35) in $N+1$ unknowns, namely N unknown values of z_i and the unknown constant R. This transformation has place if inequality (30) holds, which since $y_k = \cos\Delta x_k$ (where $\Delta x_k = x_k \cdot \Delta t$, $x_k = \omega_k \pm i\beta_k$) can be written as follows: $\cos x_k \Delta t \neq \cos x_m \Delta t$ ($k \neq m$, $k = 1, 2, \ldots, N$, $m = 1, 2, \ldots, N$) or, otherwise:

$$\Delta t \neq \frac{2\pi}{x_k \pm x_m},\ x_k \neq x_m\ \left(m = 0, 1, 2, ...;\ k \neq m;\ k = 1, 2, ..., N\right), \tag{36}$$

that is the analytical form of measurements (4) does not contain same frequency sinusoids and the measurement step Δt must satisfy inequality (36).

In order to determine all the unknowns (N values of z_i and the constant R) from system (35) it needs to be expanded to contain $N+1$ equation. This can be done by using equation (35) N times, shifting the conditional midpoint by one step Δt to the right each next time, i.e.:

$$\sum_{i=1}^{N} z_i \varphi_{i,S} = -\varphi_{0,S}\ \left(S = 0, 1, 2, ..., N-1\right),$$

where $\varphi_{i,0} \equiv \varphi_i$, $\varphi_{0,0} \equiv \varphi_0$ from (35); $\varphi_{i,S} = \delta_{N+1-i+S}$, $\varphi_{0,S} = \delta_{N+1+S}^{(N)}$, $\delta_m^{(N)}$ are given by (25), z_i is given by (27).

Hence, the expanded system of equations for identifying N unknowns z_i and R is written in the form:

$$\sum_{i=1}^{N} z_i \varphi_{i,S} = -\varphi_{0,S}\ \left(S = 0, 1, 2, ..., N-1\right);\ \sum_{i=1}^{N} z_i \left(Q_i - 2^N R\right) = -\left(Q - 2^N R\right). \tag{37}$$

All the values z_i can be found from the first N equations of system (37) in case the system determinant is non-zero:

$$D = \begin{vmatrix} \varphi_{1,0} & \varphi_{2,0} & \cdots & \varphi_{N,0} \\ \varphi_{1,1} & \varphi_{2,1} & \cdots & \varphi_{N,1} \\ \ldots\ldots & \ldots\ldots. & \ldots & \ldots\ldots.. \\ \varphi_{1,N-1} & \varphi_{2,N-1} & \cdots & \varphi_{N} \end{vmatrix} \neq 0, \tag{38}$$

which has place for the actual measurements.

Then from the last equation of (37) we find R :

$$R = \frac{\sum_{i=1}^{N} z_i Q_i + Q_0}{2^N \left(\sum_{i=1}^{N} z_i + 1\right)}. \tag{39}$$

This concludes the solution of the main problem of the method of equidistant points.

Apparently, to solve the main problem by formulae (37), (38) one needs the measurement $\alpha(t)$ taken at $3N+2$ points, where $N = 2n_1 + n_2$ according to (15), i.e. the number of measurements needs to be greater than the number of the unknowns in the measurement analytical formula (4).

The sufficient number of measurements is $N_1 = 3N + 2$, and the measurement corresponding to the conditional midpoint satisfies $p' \geq N + 2$.

The solution to the secondary problem can be derived by determining z_i from (37) and subsequent computation of y_j ($j = 1, 2, 3, ..., N$), which are in fact the roots of the equation:

$$y^N + \sum_{i=1}^{N} (-1)^i z_i y^{N-1} = 0\ \ [4].$$

The values of y_j being determined we then find $\Delta x_j = x_j \Delta t$ and can compute β_k and ω_k by formulae (10). We also note that A_k and B_k can be identified form (20) and (31), and their values can then be used to compute the amplitudes A'_k and phases φ_k of the sinusoids in (4). Since this problem is not in the focus of the present paper we limit ourselves here to just outlining the ways of it's solving.

Prospects of further research. The further development of the methods of identification of system parameters comprises:

– development of the algorithms for solving the main problem in the case when the process is given in the form:

$$\alpha = R + \sum_{k=1}^{N} A'_k(t) e^{-\beta_k t} \sin\left(\omega_k t + \psi_k\right),$$

where $A'_k(t)$ is the polynomial corresponding to the multiple roots of the differential equation considered;

– development of the efficient methods for solving the secondary problem;

– improvement of the algorithms aimed at reducing the number of the measurements required;

– construction of the error correction algorithms providing the improvement in accuracy while reducing the information retrieval time.

Studying these points provides a sufficient solution for a wide range of practical problems.

Conclusions. In this paper a mathematical substantiation and computational formulae of an innovative finite-step computational method are presented. The algorithm developed is efficient for solving both main and secondary problems. When solving such problems for specific systems their respective features need to be taken into account, which essentially improves the accuracy of the results as well as the processing time. The results and computational formulae proposed can be used for solving similar problems of identification of various automatic control systems.

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ТЕОРІЯ ПОБУДОВИ ОПЕРАТОРІВ ІНТЕРПОЛЯЦІЇ ІЗ ЗАДАНИМИ ПРОЕКЦІЯМИ

Оператори апроксимації функції двох змінних, що інтерполюють її своїми проекціями по M непаралельних прямих, недостатньо досліджувалися в науковій літературі. У той же час ця теоретична проблема викликає практичний інтерес, коли дані проекцій (інтеграли вздовж ліній)виходять із компактного сканера томографії. У роботі побудований оператор інтерполяції, який точно відновлює поліноми степеня $M-1$. Метод досліджувався для випадку системи взаємно перпендикулярних прямих та для трьох непаралельних перетинних прямих (сторін трикутника). Знайдено інтегральне представлення залишкового члена наближення диференційовних функцій отриманими операторами. Запропонований метод дозволяє розширити теорію та практичне застосування комп'ютерної томографії.

Ключові слова: комп'ютерна томографія, оператори інтерполяції з відомими проекціями, залишок наближення, апроксимація, проекції вздовж ліній.

Е. Л. ХУРДЕЙ

ТЕОРИЯ ПОСТРОЕНИЯ ОПЕРАТОРОВ ИНТЕРПОЛЯЦИИ С ЗАДАНЫМИ ПРОЕКЦИЯМИ

Операторы аппроксимации функции двух переменных, которые интерполируют ее своими проекциями по M непараллельных прямых, недостаточно исследовались в научной литературе. В то же время эта теоретическая проблема вызывает практический интерес, когда данные проекции (интегралы вдоль линий) выходят из компактного сканера томографии. В работе построен оператор интерполяции, который точно восстанавливает полином степени $M-1$. Метод исследовался для случая системы взаимно перпендикулярных прямых и для трех непараллельных пересекающихся прямых (сторон треугольника). Найдено интегральное представление остаточного члена приближения дифференцируемых функций полученными операторами. Предложенный метод позволяет расширить теорию и практическое применение компьютерной томографии.

Ключевые слова: компьютерная томография, операторы интерполяции с известными проекциями, остаток приближения, аппроксимация, проекции вдоль линий.