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Nonlinear Dynamics of a Spinning Shaft with Non-Constant Rotating Speed

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Abstract
Reported research on spinning shafts is mostly restricted to cases of constant rotational speed without examining the dynamics thatoccursduringtheir spin-up or spin-down operation. In suchcases, the motion is described by a nonlinear system of Partial Differential Equations (PDEs) coupled with an Integro-Differential Equation (IDE). The nonlinear system of PDEs with IDE, projected onto the infinite basis of the modes of the underlying linear system,results in a system of nonlinear Ordinary Differential Equations (ODEs). In this articleis appliedthe multiple scales perturbation method for dynamic analysis and the system in first order approximation takes the form of two coupled sets of pairedequations. The first pair describes torsional and rigid body rotation whilst the secondconsists of the equations describing the two lateral bending motions. Although in this system non-conservative forces are not considered in terms of damping or explicit externally applied load (torques/forces), the solution of the 1st order approximation of the first set of equations indicates that there are no periodic motions. The solution of the second set of equations of 1st order approximation coincides with the case of constant rotating speed It isshown, that the Normal Modes in bending motions are the critical speeds of the shaft. It is shown that the frequencies in the Campbell diagram coincide with the frequencies associated with the 1st order solution of the nonlinear system. Moreover, the analytical solution of the first pair of equations is in good agreement with direct numerical simulations.This work paves the way for the development of the Nonlinear Campbell diagram that can be used to determine the dynamic behaviour of rotating structures during spin-up or spin-down operation.

Keywords
Non-constant rotating speed, spin-up, spin-down, normal modes, Campbell diagram

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Introduction

Although much work has been reported about spinning shafts,only few articles are related in examining their dynamics during spin-up and spin-down operation. Suherman, and Plaut [1] developed a model and examined dynamics for a spinning shaft with non-constant rotating speed and flexible internal support, but torsional motion was neglected in their treatment. In [2], Kirk et al. developed a model for a spinning shaft with eccentric sleeves as dynamic boundary conditions considering also non-constant rotating speed, whilst Georgiades [3] projected the dynamics of the system of equations (PDEs) developed in [2] in the infinite linear modes of the underlying linear system to obtain the discrete system,and then,in case of constant rotating speed, the correlation of the eigenvalues of the discrete systemwith those obtained from Finite Element Analysis (FEA) was examined.

In this article,the method of multiple scales is used to solve the 1st order approximation of the discretized nonlinear system. The eigenvalues of the 1st order solution for bendingarethen compared witha Campbell diagram arising fromthe FEA solution in [3]. Finally,the analytical solution is compared with direct numerical simulations.

1. Equations of motion, multiple scales approach

In [2] the equations of motion (PDEs) of the spinning shaft with non-constant rotating speed considering Euler-Bernoulli beam including rotary inertia terms, made of isotropic material without including any explicit non-conservative terms such as viscous damping or externally applied load (torque or force), were derived, and they are given by,

$$(I_5+I_6)L\ddot{\theta}+\boldsymbol{\ddot{\theta}}\int_0^L\left[I_1\boldsymbol{w^2}+I_1\boldsymbol{v^2}+(I_5+I_6)\boldsymbol{\phi^2}\right]\boldsymbol{dx}-\int_0^L\left[(I_5+I_6)\ddot{\phi}\right]dx+$$
$$+\boldsymbol{2\dot{\theta}}\int_0^L\left[(I_5+I_6)\boldsymbol{\dot{\phi}\phi}+I_1\boldsymbol{v\dot{v}}+I_1\boldsymbol{w\dot{w}}\right]\boldsymbol{dx}-\int_0^L\left[I_1\boldsymbol{\ddot{w}v}-I_1\boldsymbol{\ddot{v}w}\right]\boldsymbol{dx}=0, \tag{1a}$$
$$I_1\boldsymbol{\dot{\theta}^2v}-I_1\boldsymbol{\ddot{\theta}}w-2I_1\boldsymbol{\dot{\theta}\dot{w}}-I_1\ddot{v}+(I_5\ddot{v}')'-(k_5v'')''=0, \tag{1b}$$
$$I_1\boldsymbol{\dot{\theta}^2w}+I_1\boldsymbol{\ddot{\theta}v}+2I_1\boldsymbol{\dot{\theta}\dot{v}}-I_1\ddot{w}+(I_6\ddot{w}')'-(k_6w'')''=0, \tag{1c}$$
$$(I_5+I_6)\boldsymbol{\dot{\theta}^2}+(I_5+I_6)\ddot{\theta}-(I_5+I_6)\ddot{\phi}+\left[\left(\frac{k_7+k_8}{2}\right)\phi'\right]'=0. \tag{1d}$$

The first equation describes the rigid body rotational motion ($\theta(t)$); the seconddescribes the bending motion in the y-direction ($v(s,t)$); the thirddescribes the bending motion in the z-direction ($w(s,t)$); and finally, the fourthdescribes the torsional motion ($\phi(s,t)$). It should be noted that the equation describing the axial motion is fully decoupled. In the above, the nonlinearterms are indicated in bold type, and collectively form a nonlinear system of PDEs coupled through the non-constant rotating speed with one integro-differential equationthat describes the rigid body motion of the shaft. It is readilyverified that in case of constant rotating speed this system takes the form of one decoupled torsional equation and the lateral bending equations coupled with the rotating speed as a parameter. The Boundary Conditions (B.C.s)are,

$$v(0,t)=v(L,t)=0,\ v''(0,t)=v''(L,t)=0,\ w(0,t)=w(L,t)=0, \tag{2a-f}$$
$$w''(0,t)=w''(L,t)=0,\ \phi(0,t)=0, \phi'(L,t)=0, \tag{2h-j}$$

whereas, eq. (2a-b,2e-f,2j) are the strong B.C.s arising from the geometry of the problem (simply supported shaft in bending and fixed-free torsional motion) and eq. (2c-d,2h-j) are the weak B.C.s arising from the equilibriums in free motions through the Extended Hamilton's Principle formulation. The coefficients are given by,

$$I_1=\pi\rho_0({r_o}^2-{r_i}^2),\ I_5=I_6=\rho_0 I=\pi\rho_0\left(\frac{{r_o}^4-{r_i}^4}{4}\right),\ k_5=k_6=EI, k_7=k_8=2GI, \tag{3a-d}$$

with r_o,r_i, external and internal radius of shaft respectively, L is the length of the shaft, ρ_0, E, G are the density Young's and shear modulus, respectively.

To project the system of equations (1) in the infinite base of the linear modes of the associated linear system, in [3] it was considered that for equations (1b-c) the associated linear problem is the simply supported Euler-Bernoulli beam in bending, and for the homogeneous case for the equivalent torsional equation(1d), it is the rod in axial vibration. In [3],the displacements were expressed by truncating the series into the first linear mode, then, with multiplication of each equation with each associated mode shape and then integration in longitudinal x-direction, lead to the following system of equations,

$$\left[1+\frac{q_v^2}{(2I_5L)}+\frac{q_w^2}{(2I_5L)}+\frac{q_\phi^2}{(2I_5L)}\right]\ddot{\theta}-\frac{F}{(2I_5L)}\ddot{q}_\phi-\frac{q_v\ddot{q}_w}{(2I_5L)}+\frac{\ddot{q}_vq_w}{(2I_5L)}=-\frac{\dot{\theta}\dot{q}_vq_v}{(I_5L)}-\frac{\dot{\theta}\dot{q}_wq_w}{(I_5L)}-\frac{\dot{\theta}\dot{q}_\phi q_\phi}{(I_5L)}, \tag{4a}$$
$$\ddot{\theta}q_w+(1-M)\ddot{q}_v=\left[\dot{\theta}^2-\omega_b^2(1-M)\right]q_v-2\dot{\theta}\dot{q}_w, \tag{4b}$$
$$-\ddot{\theta}q_v+(1-M)\ddot{q}_w=\left[\dot{\theta}^2-\omega_b^2(1-M)\right]q_w+2\dot{\theta}\dot{q}_v, \tag{4c}$$
$$-F\ddot{\theta}+\ddot{q}_\phi=F\dot{\theta}^2-\omega_T^2q_\phi, \tag{4d}$$

with the following constants, mode shapes, and frequencies [3],

$$F=(2I_5)\int_0^L Y_1(s)ds=\frac{4}{\pi}\sqrt{I_5L}, \qquad M=I_5\int_0^L y_1''(s)y_1(s)ds=-\frac{I_5\pi^2}{I_1L^2}, \tag{5a,b}$$

$$y_1(s) = \sqrt{\frac{2}{I_1 L}} sin\left(\frac{\pi}{L}s\right),\ \omega_b = \sqrt{\frac{\pi^4 k_5}{L^2\pi^2 I_5 + L^4 I_1}},\ Y_1(s) = \sqrt{\frac{1}{I_5 L}} sin\left(\frac{\pi}{2L}s\right), \omega_T = \frac{\pi}{4L}\sqrt{\frac{(k_7+k_8)}{I_5}}. \quad \text{(5c-f)}$$

Here, the solutions of this system ofequations (4) are considered in the following form,

$$\theta = \varepsilon^0\theta_0 + \varepsilon^1\theta_1 + \varepsilon^2\theta_2 + H.O.T., \qquad q_v = \varepsilon^1 q_{v,1} + \varepsilon^2 q_{v,2} + H.O.T., \quad \text{(6a-b)}$$
$$q_w = \varepsilon^1 q_{w,1} + \varepsilon^2 q_{w,2} + H.O.T., \qquad q_\phi = \varepsilon^1 q_{\phi,1} + \varepsilon^2 q_{\phi,2} + H.O.T. \quad \text{(6c-d)}$$

Also, following the multiple scales approach, the system of equations (4) takes the form of the various ε-scale orders:

$\underline{\varepsilon^0}$,

$$D_0^2\theta_0 = 0 \Leftrightarrow D_0\theta_0 = \Omega \Leftrightarrow \theta_0 = \Omega T_0 + ct, \quad (7)$$

$\underline{\varepsilon^1}$,

$$2I_5 L D_0^2\theta_1 - F D_0^2 q_{\phi,1} = -4I_5 L D_0 D_1\theta_0, \quad \text{(8a)}$$
$$D_0^2\theta_0 q_{w,1} + (1-M)D_0^2 q_{v,1} - (D_0\theta_0)^2 q_{v,1} + \omega_b^2(1-M)q_{v,1} + 2D_0\theta_0 D_0 q_{w,1} = 0, \quad \text{(8b)}$$
$$-D_0^2\theta_0 q_{v,1} + (1-M)D_0^2 q_{w,1} - (D_0\theta_0)^2 q_{w,1} + \omega_b^2(1-M)q_{w,1} - 2D_0\theta_0 D_0 q_{v,1} = 0, \quad \text{(8c)}$$
$$-F D_0^2\theta_1 + D_0^2 q_{\phi,1} - 2F D_0\theta_0 D_0\theta_1 + \omega_T^2 q_{\phi,1} = 2F D_0 D_1\theta_0 + 2F D_0\theta_0 D_1\theta_0, \quad \text{(8d)}$$

$\underline{\varepsilon^2}$,

$$\begin{aligned} &2I_5 L D_0^2\theta_2 - F D_0^2 q_{\phi,2} = F_1 = \\ &= -2I_5 L(2D_0 D_1\theta_1 + 2D_0 D_2\theta_0 + D_1^2\theta_0) - D_0^2\theta_0 q_{v,1}^2 - D_0^2\theta_0 q_{w,1}^2 - D_0^2\theta_0 q_{\phi,1}^2 + 2F D_0 D_1 q_{\phi,1} + \\ &+ q_{v,1} D_0^2 q_{w,1} - q_{w,1} D_0^2 q_{v,1} - 2D_0\theta_0 D_0 q_{v,1} q_{v,1} - 2D_0\theta_0 D_0 q_{w,1} q_{w,1} - 2D_0\theta_0 D_0 q_{\phi,1} q_{\phi,1}, \end{aligned} \quad \text{(9a)}$$

$$\begin{aligned} &D_0^2\theta_0 q_{w,2} + (1-M)D_0^2 q_{v,2} - (D_0\theta_0)^2 q_{v,2} + \omega_b^2(1-M)q_{v,2} + 2D_0\theta_0 D_0 q_{w,2} = F_2 = \\ &= D_0^2\theta_1 q_{w,1} - 2D_0 D_1\theta_0 q_{w,1} - 2(1-M)D_0 D_1 q_{v,1} + 2D_0\theta_0 D_0\theta_1 q_{v,1} + \\ &+ 2D_0\theta_0 D_1\theta_0 q_{v,1} - 2D_0\theta_0 D_1 q_{w,1} - 2D_0\theta_1 D_0 q_{w,1} - 2D_1\theta_0 D_0 q_{w,1}, \end{aligned} \quad \text{(9b)}$$

$$\begin{aligned} &-D_0^2\theta_0 q_{v,2} + (1-M)D_0^2 q_{w,2} - (D_0\theta_0)^2 q_{w,2} + \omega_b^2(1-M)q_{w,2} - 2D_0\theta_0 D_0 q_{v,2} = F_3 = \\ &= D_0^2\theta_1 q_{v,1} + 2D_0 D_1\theta_0 q_{v,1} - 2(1-M)D_0 D_1 q_{w,1} + 2D_0\theta_0 D_0\theta_1 q_{w,1} + \\ &+ 2D_0\theta_0 D_1\theta_0 q_{w,1} + 2D_0\theta_0 D_1 q_{v,1} + 2D_0\theta_1 D_0 q_{v,1} + 2D_1\theta_0 D_0 q_{v,1}, \end{aligned} \quad \text{(9c)}$$

$$\begin{aligned} &-F D_0^2\theta_2 + D_0^2 q_{\phi,2} - 2F D_0\theta_0 D_0\theta_2 + \omega_T^2 q_{\phi,2} = F_4 = \\ &= F(2D_0 D_1\theta_1 + 2D_0 D_2\theta_0 + D_1^2\theta_0) - 2D_0 D_1 q_{\phi,1} + \\ &+ F[(D_0\theta_1)^2 + (D_1\theta_0)^2 + 2D_0\theta_1 D_1\theta_0 + 2D_0\theta_0 D_1\theta_1 + 2D_0\theta_0 D_2\theta_0]. \end{aligned} \quad \text{(9d)}$$

It is notable that the left sides of equations for 1[st]and 2[nd]order equations are not fully coupled, but are coupled in pairs. The first pair consists of theequation for the rigid body rotation with the torsional motion and the other with the two equations for the lateral bending motions.

2. Analytical solutions

2.1 Analytical solution of 1[st] order approximation, for torsional-rigid body rotation, motions

Here, the solution of the 1[st]order approximation for the rigid body motion coupled with torsion, is considered.Elimination of the secular terms in equation (8a) and taking into consideration equation (7) leads to,

$$D_0 D_1\theta_0 = 0 \Leftrightarrow D_1\Omega = 0 \Leftrightarrow D_1\Omega T_0 = D_1\theta_0 = 0. \quad (10)$$

Considering (10), all secular terms in the right-hand side of equation (8d) are seen to be eliminated.To simplify the equations for the rest of the article over-dot notation will be used instead of D_0, and dash notation will be used instead of D_1.

The 1st order approximation for rigid body rotation (eq. 8a) with torsion (eq. 8d) considering equations(7,10) and the new notation, can be written in the following form,

$$2I_5L\ddot{\theta}_1 - F\ddot{q}_{\phi,1} = 0, \tag{11a}$$

$$-F\ddot{\theta}_1 + \ddot{q}_{\phi,1} - 2F\Omega\dot{\theta}_1 + \omega_T^2 q_{\phi,1} = 0. \tag{11b}$$

In the above system (11) the angular rigid body position is involved only with its derivative,thereforethis system can be solved with respect to $\dot{\theta}_1$. Then, the angular position can be trivially obtained by integration in time of the angular velocity. The system of equations (11) can be solved by writing asa first order system of 3 differential equations with respect to$\dot{\theta}_1$, $q_{\phi,1}$, $\dot{q}_{\phi,1}$. The eigenvalues of this system, in case of,

$$2I_5L(2I_5L - F^2)\omega_T^2 > F^4\Omega^2. \text{ (relative small rotating speed)} \tag{12}$$

They,are given by,

$$\lambda_{1,1} = 0, \lambda_{1,2\div3} = \zeta_0 \pm i\mu_0, \tag{13a-c}$$

with,

$$\zeta_0 = \frac{F^2\Omega}{2I_5L-F^2}, \quad \mu_0 = \sqrt{\frac{\omega_T^2 2I_5L(2I_5L-F^2)-F^4\Omega^2}{(2I_5L-F^2)^2}}. \tag{14a,b}$$

The real part of eigenvalues can be zero($\zeta_0 = 0$) only in the case of non-rotation with$\Omega = 0$. *Therefore, although a non-conservative load in explicit form is not included,the spinning shaft with non-constant rotating speed has no periodic motions* since the real part of the eigenvalues is never zero. It should be also highlighted, that the imaginary part of the eigenvalues (μ_0) which define the frequencies of the motions are very different from the natural frequencies in torsion (ω_T).

The solution of the corresponding 1st orderapproximation system (eq. 11) is given by,

$$\begin{Bmatrix} \dot{\theta}_1(T_0) \\ q_{\phi,1}(T_0) \\ \dot{q}_{\phi,1}(T_0) \end{Bmatrix} = [P_1]\left[diag\left(exp\left(\lambda_{1,j}T_0\right)\right)\right][P_1^{-1}] \begin{Bmatrix} \dot{\theta}_1(0) \\ q_{\phi,1}(0) \\ \dot{q}_{\phi,1}(0) \end{Bmatrix}, \qquad j = 1,2,3 \tag{15}$$

whereas,P_1 is the matrix with the associated eigenvectors. To simplify the expressions,the following constants are defined,

$$a_1 = \frac{2F\Omega}{\omega_T^2}, b_1 = \frac{2I_5L}{F}, c_1 = \frac{2I_5L\zeta_0}{F(\zeta_0^2+\mu_0^2)}, d_1 = \frac{-2I_5L\mu_0}{F(\zeta_0^2+\mu_0^2)}, g_1 = \frac{a_1}{2d_1} = \frac{-F^2\Omega(\zeta_0^2+\mu_0^2)}{2I_5L\mu_0\omega_T^2}, \tag{16a-e}$$

$$f_1 = \frac{1}{2d_1} = \frac{-F(\zeta_0^2+\mu_0^2)}{4I_5L\mu_0}, h_1 = \frac{1}{2b_1} = \frac{F}{4I_5L}, k_1 = \frac{a_1-c_1}{2b_1d_1} = \frac{I_5L\zeta_0F - F\Omega(\zeta_0^2+\mu_0^2)}{4(I_5L)^2\mu_0}. \tag{16f-h}$$

Then the matrix of eigenvectors(P_1)and its inverse (P_1^{-1})are given by,

$$P_1 = \begin{bmatrix} 1 & 1 & 1 \\ a_1 & c_1 + id_1 & c_1 - id_1 \\ 0 & b_1 & b_1 \end{bmatrix}, \qquad P_1^{-1} = \begin{bmatrix} 1 & 0 & -1/b_1 \\ ig_1 & -if_1 & h_1 - ik_1 \\ -ig_1 & if_1 & h_1 + ik_1 \end{bmatrix}. \tag{17a,b}$$

Now, from equation(15), and taking into consideration equations (13-14,16-17) the explicit form solution of the 1st order approximation of the system of equations (11)is given by,

$$\dot{\theta}_1(T_0) = A_{11} + e^{\zeta_0T_0}\left(2A_{\theta,1}cos(\mu_0T_0) + 2A_{\theta,2}sin(\mu_0T_0)\right) = A_{11} + A_{12}e^{(\zeta_0+i\mu_0)T_0} + cc, \tag{18a}$$

$$q_{\phi,1}(T_0) = A_{21} + e^{\zeta_0 T_0}\left(2A_{\phi,1}cos(\mu_0 T_0) + 2A_{\phi,2}sin(\mu_0 T_0)\right) = A_{21} + A_{22}e^{(\zeta_0+i\mu_0)T_0} + cc, \quad (18b)$$

$$\dot{q}_{\phi,1}(T_0) = 2e^{\zeta_0 T_0}\left[\left(\zeta_0 A_{\phi,1} + \mu_0 A_{\phi,2}\right)cos(\mu_0 T_0) + \left(\zeta_0 A_{\phi,2} - \mu_0 A_{\phi,1}\right)sin(\mu_0 T_0)\right], \quad (18c)$$

with,

$$A_{11} = \dot{\theta}_1(0) - \frac{\dot{q}_{\phi,1}(0)}{b_1}, \quad A_{\theta,1} = h_1\dot{q}_{\phi,1}(0), \quad A_{\theta,2} = -g_1\dot{\theta}_1(0) + f_1 q_{\phi,1}(0) + k_1\dot{q}_{\phi,1}(0), \quad (19a-c)$$

$$A_{21} = a_1\dot{\theta}_1(0) - \frac{a_1}{b_1}\dot{q}_{\phi,1}(0), \quad A_{\phi,1} = -d_1 g_1\dot{\theta}_1(0) + d_1 f_1 q_{\phi,1}(0) + (c_1 h_1 + d_1 k_1)\dot{q}_{\phi,1}(0), \quad (19d-e)$$

$$A_{\phi,2} = -c_1 g_1\dot{\theta}_1(0) + c_1 f_1 q_{\phi,1}(0) - (h_1 d_1 - c_1 k_1)\dot{q}_{\phi,1}(0), \quad (19f)$$

$$A_{12} = A_{\theta,1} - iA_{\theta,2}, \quad A_{22} = A_{\phi,1} - iA_{\phi,2}. \quad (19f-h)$$

2.2 Analytical solution of 1st order approximation for lateral bending motions

In this section, the second pair of differential equations (8b-c) of 1storder approximation, are solved. Usingthe new notation, and considering (7,10),the 1storder approximation of equations of motion for lateral bending (eq. 8b-c) take the form,

$$(1-M)\ddot{q}_{v,1} - \Omega^2 q_{v,1} + \omega_b^2(1-M)q_{v,1} + 2\Omega\dot{q}_{w,1} = 0, \quad (20a)$$

$$(1-M)\ddot{q}_{w,1} - \Omega^2 q_{w,1} + \omega_b^2(1-M)q_{w,1} - 2\Omega\dot{q}_{v,1} = 0. \quad (20b)$$

The system of equations (20) is the same as one describing the motion of a spinning shaft in case of constant rotating speed [3], and the solution can be obtained by writing the system as first order differential equations.Consider now relative small rotating speeds which obey the following two conditions,

$$\Omega^2 < -\frac{(1-M)^2\omega_b^2}{M}, \qquad -\omega_b^2 - \frac{(M+1)}{(1-M)^2}\Omega^2 + \frac{2\Omega}{(1-M)}\sqrt{\frac{M\Omega^2}{(1-M)^2} + \omega_b^2} < 0. \quad (21a,b)$$

Then, the eigenvalues of this system (eq. 20) are,

$$\lambda_{2,1} = -i\omega_1, \quad \lambda_{2,2} = -i\omega_2, \quad \lambda_{2,3} = i\omega_1, \quad \lambda_{2,4} = i\omega_2, \quad (22a-d)$$

withthe explicit form of the natural frequencies with respect to rotating speed given by,

$$\omega_{1\div 2} = \sqrt{\omega_b^2 + \frac{(M+1)}{(1-M)^2}\Omega^2 \mp \frac{2\Omega}{(1-M)}\sqrt{\frac{M\Omega^2}{(1-M)^2} + \omega_b^2}}. \quad (23)$$

A plot of these frequencies, from (23),with respect to the rotating speed is used to form the Campbell diagram for a shaft with a constant rotating speed. It should be noted that in case of neglecting the rotary inertia terms in bending, with$M = 0$, and (23) can then be simplified to the form,

$$\omega_{1\div 2} = |\omega_b \mp \Omega|. \quad (24)$$

It should be noted, that based on previously reported research on nonlinear dynamics (also herein for the other set of equations), in the case that secular terms in ε^2 are eliminated,then the frequencies defined by equations (23,24) are expected to have detuning frequencies in T_1 scale, and therefore these frequencies ($\omega_{1,2}$) will no longer be the actual 'resonant' frequencies in bending during spin-up or spin-down.

In case of constant rotating speed, it is notable that based on the latest definition of the Normal Modes which are the**periodic motions**, not all frequencies are associated with the Normal Modes since the periodicity condition for the angular position must satisfy,

$$\theta_0(T_{0,T}) - \theta_{0,0} = mod\left[\Omega T_{0,T}, 2\pi\right] = mod\left[\frac{2\pi\Omega}{\omega_{1,2}}, 2\pi\right] = 0, \quad (25)$$

andthis is true only when $\omega_j = n\Omega$(with j=1,2 and n any integer). *For n=1 then it is the 1-1 resonance which is the case of Critical Speeds of the shaft and these are the frequencies of the associated Normal Modes of the shaft*, which justifies the resonances in FRFs of imbalanced shaftswhen the rotating speeds are near (due to damping) to the critical speeds.
The solution of the system of equations (20) is given by,

$$\begin{Bmatrix} \dot{q}_{v,1}(T_0) \\ \dot{q}_{w,1}(T_0) \\ q_{v,1}(T_0) \\ q_{w,1}(T_0) \end{Bmatrix} = [P_2]\left[diag\left(exp\left(\lambda_{2,j}T_0\right)\right)\right][P_2^{-1}]\begin{Bmatrix} \dot{q}_{v,1}(0) \\ \dot{q}_{w,1}(0) \\ q_{v,1}(0) \\ q_{w,1}(0) \end{Bmatrix}, \quad j = 1,2,3,4 \quad (26)$$

The matrix of the associated eigenvectors (P_2), and its inverse (P_2^{-1})are given by,

$$P_2 = \begin{bmatrix} -i\omega_1, -i\omega_2, i\omega_1, i\omega_2 \\ -b_2, \ -d_2, -b_2, -d_2 \\ 1, \quad 1, \quad 1, \quad 1 \\ -i\frac{b_2}{\omega_1}, -i\frac{d_2}{\omega_2}, i\frac{b_2}{\omega_1}, i\frac{d_2}{\omega_2} \end{bmatrix}, \quad P_2^{-1} = \begin{bmatrix} -i\omega_1 d_2 d_{n1}, & -d_{n2}, & -d_2 d_{n2}, & i\omega_1\omega_2^2 d_{n1} \\ i\omega_2 b_2 d_{n1}, & d_{n2}, & b_2 d_{n2}, & -i\omega_2\omega_1^2 d_{n1} \\ i\omega_1 d_2 d_{n1}, & -d_{n2}, & -d_2 d_{n2}, & -i\omega_1\omega_2^2 d_{n1} \\ -i\omega_2 b_2 d_{n1}, & d_{n2}, & b_2 d_{n2}, & i\omega_2\omega_1^2 d_{n1} \end{bmatrix}, \quad (27a,b)$$

with,

$$b_2 = \frac{-\Omega^2 + (1-M)(\omega_b^2 - \omega_1^2)}{2\Omega}, \qquad d_2 = \frac{-\Omega^2 + (1-M)(\omega_b^2 - \omega_2^2)}{2\Omega}, \quad (28a,b)$$

$$d_{n1} = \frac{\Omega}{[-\Omega^2 + (1-M)(\omega_b^2 - \omega_1^2)]\omega_2^2 - [-\Omega^2 + (1-M)(\omega_b^2 - \omega_2^2)]\omega_1^2}, \quad d_{n2} = \frac{\Omega}{(1-M)(\omega_2^2 - \omega_1^2)}. \quad (28c,d)$$

Therefore, considering equations (22-23,26-28) the solutions of equations (20) are given by,

$$q_{v,1}(T_0) = C_{v1}e^{i\omega_1 T_0} + D_{v1}e^{i\omega_2 T_0} + cc, \quad (29a)$$
$$q_{w,1}(T_0) = C_{w1}e^{i\omega_1 T_0} + D_{w1}e^{i\omega_2 T_0} + cc, \quad (29b)$$

with,

$$C_{v1} = B_{v1,1} + iB_{v2,1}, D_{v1} = B_{v1,2} + iB_{v2,2}, C_{w1} = B_{w1,1} + iB_{w2,1}, D_{w1} = B_{w1,2} + iB_{w2,2}, \quad (30a\text{-}d)$$
$$B_{v1,1} = -d_{n2}\dot{q}_{w,1}(0) - d_{n2}d_2 q_{v,1}(0), \qquad B_{v1,2} = d_{n2}\dot{q}_{w,1}(0) + d_{n2}b_2 q_{v,1}(0), \quad (30e\text{-}f)$$
$$B_{v2,1} = d_{n1}\omega_1 d_2 \dot{q}_{v,1}(0) - d_{n1}\omega_1\omega_2^2 q_{w,1}(0), B_{v2,2} = -d_{n1}b_2\omega_2\dot{q}_{v,1}(0) + d_{n1}\omega_1^2\omega_2 q_{w,1}(0), \quad (30g\text{-}h)$$
$$B_{w1,1} = -d_{n1}b_2 d_2 \dot{q}_{v,1}(0) + d_{n1}b_2\omega_2^2 q_{w,1}(0), B_{w1,2} = d_{n1}b_2 d_2\dot{q}_{v,1}(0) - d_{n1}\omega_1^2 d_2 q_{w,1}(0), \quad (30i\text{-}j)$$
$$B_{w2,1} = -d_{n2}\left(\frac{b_2}{\omega_1}\right)\dot{q}_{w,1}(0) - d_{n2}(b_2 d_2/\omega_1)q_{v,1}(0), \quad (30k)$$
$$B_{w2,2} = d_{n2}\left(\frac{d_2}{\omega_2}\right)\dot{q}_{w,1}(0) + d_{n2}(b_2 d_2/\omega_2)q_{v,1}(0). \quad (30l)$$

2.3 Solution of amplitude modulation equations for rigid body and torsional motions

In, order to finalise the 1storder approximation solution for rigid body and torsional motions, on this section the amplitude constants in the equations (19) with respect to time scaleT_1, we will be determined by eliminating the secular terms of ε^2 in equations(9a,9d) and solving the amplitude modulation equations. Considering equations (7,10)and elimination of T_2secular terms, then the right-hand side of equations (9a,9d)lead to,

$$D_2\Omega = 0 \Leftrightarrow D_2\Omega T_0 = 0 \Leftrightarrow D_2\theta_0 = 0, \quad (31)$$
$$F_1 = -4I_5 L D_1\dot{\theta}_1 + 2FD_1\dot{q}_{\phi,1} - 2\Omega\dot{q}_{\phi,1}q_{\phi,1} + q_{v,1}\ddot{q}_{w,1} - q_{w,1}\ddot{q}_{v,1} - 2\Omega\dot{q}_{v,1}q_{v,1} - 2\Omega\dot{q}_{w,1}q_{w,1}, \quad (32a)$$
$$F_4 = 2FD_1\dot{\theta}_1 - 2D_1\dot{q}_{\phi,1} + F\left(\dot{\theta}_1\right)^2 + 2F\Omega D_1\theta_1. \quad (32b)$$

Considering the solutions of first order approximation (18) after some manipulations and using the new notation, thenthe equations (32) are taking the form,

$$\begin{aligned}
F_1 = -4I_5L\left[A'_{11} + e^{\zeta_0 T_0}\left(2A'_{\theta,1}cos(\mu_0 T_0) + 2A'_{\theta,2}sin(\mu_0 T_0)\right)\right] + \\
+4Fe^{\zeta_0 T_0}\left[\left(\zeta_0 A'_{\phi,1} + \mu_0 A'_{\phi,2}\right)cos(\mu_0 T_0) + \left(\zeta_0 A'_{\phi,2} - \mu_0 A'_{\phi,1}\right)sin(\mu_0 T_0)\right] - \\
-4\Omega\{A_{21}e^{\zeta_0 T_0}\left[\left(\zeta_0 A'_{\phi,1} + \mu_0 A'_{\phi,2}\right)cos(\mu_0 T_0) + \left(\zeta_0 A'_{\phi,2} - \mu_0 A'_{\phi,1}\right)sin(\mu_0 T_0)\right] + \\
+e^{2\zeta_0 T_0}\zeta_0\left(A_{\phi,1}A'_{\phi,1} + A_{\phi,2}A'_{\phi,2}\right) + \\
+2e^{2\zeta_0 T_0}\{\left[\zeta_0\left(A_{\phi,1}A'_{\phi,1} - A_{\phi,2}A'_{\phi,2}\right) + \mu_0\left(A_{\phi,2}A'_{\phi,1} + A_{\phi,1}A'_{\phi,2}\right)\right]cos(2\mu_0 T_0) + \\
+\left[\zeta_0\left(A_{\phi,2}A'_{\phi,1} + A_{\phi,1}A'_{\phi,2}\right) - \mu_0\left(A_{\phi,1}A'_{\phi,1} - A_{\phi,2}A'_{\phi,2}\right)\right]sin(2\mu_0 T_0)\}\} + \\
+q_{v,1}\ddot{q}_{w,1} - q_{w,1}\ddot{q}_{v,1} - 2\Omega\dot{q}_{v,1}q_{v,1} - 2\Omega\dot{q}_{w,1}q_{w,1},
\end{aligned} \tag{33a}$$

$$\begin{aligned}
F_4 = 2F\left[A'_{11} + e^{\zeta_0 T_0}\left(2A'_{\theta,1}cos(\mu_0 T_0) + 2A'_{\theta,2}sin(\mu_0 T_0)\right)\right] - \\
-2e^{\zeta_0 T_0}\left[2\left(\zeta_0 A'_{\phi,1} + \mu_0 A'_{\phi,2}\right)cos(\mu_0 T_0) + \right. \\
\left. +2\left(\zeta_0 A'_{\phi,2} - \mu_0 A'_{\phi,1}\right)sin(\mu_0 T_0)\right] + FA_{11}^2 + 2Fe^{2\zeta_0 T_0}\left(A_{\theta,1}^2 + A_{\theta,2}^2\right) + \\
+2Fe^{2\zeta_0 T_0}\left[\left(A_{\theta,1}^2 - A_{\theta,2}^2\right)cos(2\mu_0 T_0) + 2A_{\theta,1}A_{\theta,2}sin(2\mu_0 T_0)\right] + \\
+4Fe^{\zeta_0 T_0}\left(A_{11}A_{\theta,1}cos(\mu_0 T_0) + A_{11}A_{\theta,2}sin(\mu_0 T_0)\right) + \\
+2\Omega\left\{A'_{11}T_0 + 2e^{\zeta_0 T_0}\left[\left(\frac{A'_{\theta,1}\zeta_0 - A'_{\theta,2}\mu_0}{\zeta_0^2+\mu_0^2}\right)cos(\mu_0 T_0) + \left(\frac{A'_{\theta,1}\mu_0 + A'_{\theta,2}\zeta_0}{\zeta_0^2+\mu_0^2}\right)sin(\mu_0 T_0)\right]\right\}.
\end{aligned} \tag{33b}$$

Averaging in cosine and sine terms with frequencies $\mu_0$and elimination of the corresponding secular terms in equations (33) lead to the following amplitude modulation equations:

$$A'_{11} = 0, \tag{34}$$

$$-2I_5LA'_{\theta,1} + (F - \Omega A_{21})\zeta_0 A'_{\phi,1} + (F - \Omega A_{21})\mu_0 A'_{\phi,2} = 0, \tag{35a}$$

$$\left(F + \frac{\Omega\zeta_0}{\zeta_0^2+\mu_0^2}\right)A'_{\theta,1} - \Omega\left(\frac{\mu_0}{\zeta_0^2+\mu_0^2}\right)A'_{\theta,2} - \zeta_0 A'_{\phi,1} - \mu_0 A'_{\phi,2} = -FA_{11}A_{\theta,1}, \tag{35b}$$

$$-2I_5LA'_{\theta,2} - (F - \Omega A_{21})\mu_0 A'_{\phi,1} + (F - \Omega A_{21})\zeta_0 A'_{\phi,2} = 0, \tag{35c}$$

$$\left(\frac{\Omega\mu_0}{\zeta_0^2+\mu_0^2}\right)A'_{\theta,1} + \left(F + \frac{\Omega\zeta_0}{\zeta_0^2+\mu_0^2}\right)A'_{\theta,2} + \mu_0 A'_{\phi,1} - \zeta_0 A'_{\phi,2} = -FA_{11}A_{\theta,2}. \tag{35d}$$

Eliminating the coupling terms with derivativesin equations (35)by inversion of the matrix that multiplies theseterms and therefore pre-multiplication of the system with this inverse matrix, leads to the following system,

$$\begin{Bmatrix} A'_{\theta,1}(T_1) \\ A'_{\theta,2}(T_1) \\ A'_{\phi,1}(T_1) \\ A'_{\phi,2}(T_1) \end{Bmatrix} = \begin{bmatrix} a_{11}, & a_{12}, & 0, & 0 \\ -a_{12}, & a_{11}, & 0, & 0 \\ a_{31}, & a_{32}, & 0, & 0 \\ -a_{32}, & a_{31}, & 0, & 0 \end{bmatrix} \begin{Bmatrix} A_{\theta,1}(T_1) \\ A_{\theta,2}(T_1) \\ A_{\phi,1}(T_1) \\ A_{\phi,2}(T_1) \end{Bmatrix}, \tag{36}$$

with,

$$a_{11} = \frac{-b_3e_3(a_3 - b_3c_3)}{dn_3}, \qquad a_{12} = \frac{b_3^2d_3e_3}{dn_3}, \qquad a_{31} = \frac{a_3e_3(-a_3\zeta_0 + b_3c_3\zeta_0 + b_3d_3\mu_0)}{(\zeta_0^2+\mu_0^2)dn_3}, \tag{37a-c}$$

$$a_{32} = \frac{a_3e_3(a_3\mu_0 - b_3c_3\mu_0 + b_3d_3\zeta_0)}{(\zeta_0^2+\mu_0^2)dn_3}, \quad dn_3 = a_3^2 - 2a_3b_3c_3 + b_3^2c_3^2 + b_3^2d_3^2, \tag{37d-e}$$

$$a_3 = 2I_5L, \qquad b_3 = (F - \Omega A_{21}),\ c_3 = F + \frac{\Omega\zeta_0}{\zeta_0^2+\mu_0^2}, \qquad d_3 = \frac{\Omega\mu_0}{\zeta_0^2+\mu_0^2}, \qquad e_3 = -FA_{11}. \tag{37f-j}$$

In (36) the first two equations are fully decoupled from the rest,and their eigenvalues are given by,

$$\lambda_{3,1\div 2} = \zeta_1 \pm i\mu_1 = \frac{(b_3c_3 - a_3)e_3b_3}{(a_3 - b_3c_3)^2 + b_3^2d_3^2} \pm i\frac{b_3^2e_3d_3}{(a_3 - b_3c_3)^2 + b_3^2d_3^2}, \tag{38}$$

The solution of the first two equations of this system (eq. 36)are given by,

$$\begin{Bmatrix} A_{\theta,1}(T_1) \\ A_{\theta,2}(T_1) \end{Bmatrix} = [P_3][diag(exp(\lambda_{3,k}T_1))][P_3^{-1}]\begin{Bmatrix} A_{\theta,1}(0) \\ A_{\theta,2}(0) \end{Bmatrix}, \qquad k = 1,2, \tag{39}$$

with the eigenvectors matrix (P_3) and its inverse(P_3^{-1})given by,

$$[P_3] = \begin{bmatrix} 1 & 1 \\ \frac{\zeta_1 - a_{11} + i\mu_1}{a_{12}} & \frac{\zeta_1 - a_{11} - i\mu_1}{a_{12}} \end{bmatrix}, \qquad [P_3^{-1}] = \begin{bmatrix} \frac{\mu_1 + (\zeta_1 - a_{11})i}{2\mu_1} & -\frac{a_{12}}{2\mu_1} i \\ \frac{\mu_1 - (\zeta_1 - a_{11})i}{2\mu_1} & \frac{a_{12}}{2\mu_1} i \end{bmatrix}, \tag{40a,b}$$

Finally, considering the equations (36-40) the amplitudes for rigid-body rotation and torsional motions in the time scale T_1, are given by,

$$A_{\theta,1}(T_1) = e^{\zeta_1 T_1}\left(N_{R,1} + iN_{I,1}\right)e^{i\mu_1 T_1} + cc = e^{\zeta_1 T_1}\left[2N_{R,1}cos(\mu_1 T_1) - 2N_{I,1}sin(\mu_1 T_1)\right], \tag{41a}$$

$$A_{\theta,2}(T_1) = e^{\zeta_1 T_1}\left(N_{R,2} + iN_{I,2}\right)e^{i\mu_1 T_1} + cc = e^{\zeta_1 T_1}\left[2N_{R,2}cos(\mu_1 T_1) - 2N_{I,2}sin(\mu_1 T_1)\right], \tag{41b}$$

$$A_{\phi,1}(T_1) = \int\left[a_{31}A_{\theta,1}(T_1) + a_{32}A_{\theta,2}(T_1)\right]dT_1 + c_1 = e^{\zeta_1 T_1}\left(M_{R,1} + iM_{I,1}\right)e^{i\mu_1 T_1} + cc + c_1, \tag{41c}$$

$$A_{\phi,2}(T_1) = \int\left[-a_{32}A_{\theta,1}(T_1) + a_{31}A_{\theta,2}(T_1)\right]dT_1 + c_2 = e^{\zeta_1 T_1}\left(M_{R,2} + iM_{I,2}\right)e^{i\mu_1 T_1} + cc + c_2, \tag{41d}$$

with,

$$N_{R,1} = \frac{A_{\theta,1}(0)}{2}, \quad N_{I,1} = \frac{(\zeta_1 - a_{11})A_{\theta,1}(0) - a_{12}A_{\theta,2}(0)}{2\mu_1}, \quad N_{R,2} = \frac{A_{\theta,2}(0)}{2}, \tag{42a-c}$$

$$N_{I,2} = \frac{(\mu_1^2 + (\zeta_1 - a_{11})^2)A_{\theta,1}(0) - a_{12}(\zeta_1 - a_{11})A_{\theta,2}(0)}{2a_{12}\mu_1}, c_1 = A_{\phi,1}(0) - 2M_{R,1}, c_2 = A_{\phi,2}(0) - 2M_{R,2}, \tag{42d-f}$$

$$M_{R,1} = \frac{a_{31}(\zeta_1 N_{R,1} + \mu_1 N_{I,1}) + a_{32}(\zeta_1 N_{R,2} + \mu_1 N_{I,2})}{\zeta_1^2 + \mu_1^2}, \quad M_{I,1} = \frac{a_{31}(\zeta_1 N_{I,1} - \mu_1 N_{R,1}) + a_{32}(\zeta_1 N_{I,2} - \mu_1 N_{R,2})}{\zeta_1^2 + \mu_1^2}, \tag{42g-h}$$

$$M_{R,2} = \frac{-a_{32}(\zeta_1 N_{R,1} + \mu_1 N_{I,1}) + a_{31}(\zeta_1 N_{R,2} + \mu_1 N_{I,2})}{\zeta_1^2 + \mu_1^2}, \quad M_{I,2} = \frac{-a_{32}(\zeta_1 N_{I,1} - \mu_1 N_{R,1}) + a_{31}(\zeta_1 N_{I,2} - \mu_1 N_{R,2})}{\zeta_1^2 + \mu_1^2}. \tag{42i-j}$$

3. Numerical results

A stainless steel shaft with external and internal radius r_o=0.031 m,andr_i=0.028 m, respectively, and oflength L=1.188 m,density isρ_0=7850 Kg/m^3, and the Young's and the shear modulus are E=200 GPa, G=79.3 GPa, respectively, is considered. It should be noted that the particular shaft is thin-walled since the ratio of length with thickness is 396 (>> 10) therefore for the examination of the lower modes of vibration it can be modelled as Euler-Bernoulli beam by neglecting the shear effects [4].

3.1 Campbell diagram

In [3] the Campbell diagram of the spinning shaft with a constant rotating speed was determined through FEA. In Figure 1, the Campbell diagram obtained using equations (23) and also the data from the solution in [3] using FEA, is given, where the analytical solution is in full agreement with the results obtained from FEA.

3.2 Transient responses of 1storder approximation for rigid body rotation and torsion

In this section, the results obtained from the analytical solution of the 1storder approximation for rigid body with torsional motion, including both time scales are compared with the direct numerical integration of the original system (4).A relatively high rotating speed of 1000 rad/sec (approx. 9550 RPM) is considered, where the condition defined by (12) is still valid and this is the only nonzero initial condition.Figure 2a shows the responses for the rigid body angular velocity ($\dot{\theta}$)and shows the same trend (increasing amplitude), and for small energies the theoretical with numerical responses are in relatively good agreement, especially considering that the analytical solution is restricted to only the 1st order approximation.Figure 2b shows the responses for torsional modal angle (q_ϕ), where for small energies the numerical solutionis in relatively good agreement with the analytical solution.Finally, it is notable that for these initial conditions the numerical responses of both lateral bending motions are equal to zero, whereas the decoupling, of equations in two pairs of systems in the analytical approach, is justified.

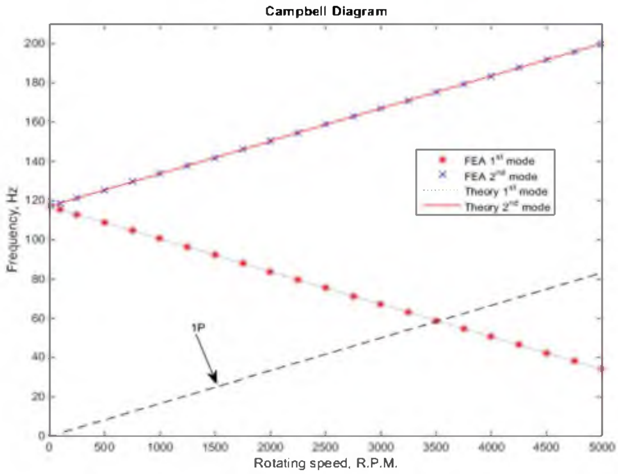

Figure 1. Campbell diagram

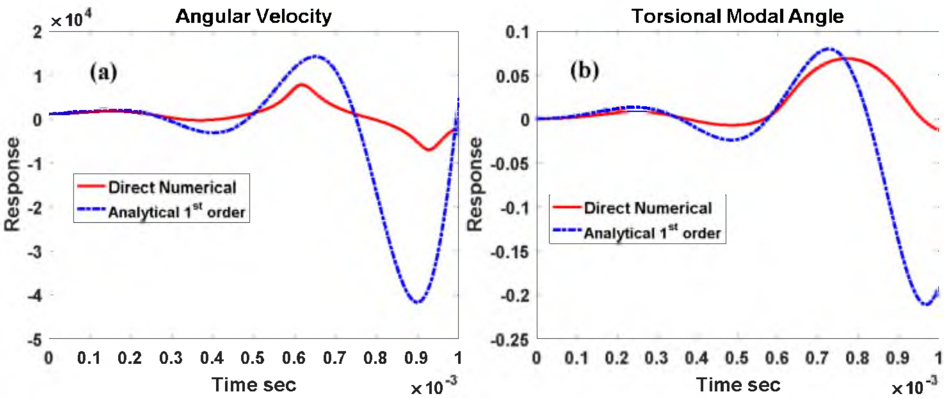

Figure 2.(a) Transient response of the angular velocity ($\dot{\theta}$).**(b)**Transient response of the torsional modal angle (q_{ϕ}).

Conclusions

In this article, the method of multiple scales is used to solve the nonlinear system which describes the motion of a spinning shaft with non-constant rotating speed. The system`s equations were written up to 2nd order time scale and their left side showed that the four originally coupled equations were coupled in pairs. The first pair consists of the equations describing rigid body motion coupled with torsion and the 1storder approximation solution showed that although, there are not included in explicit form any non-conservative forces, there are no periodic motions on this system. The comparison of this solution with numerical simulations showed relatively good agreement. The second pair of equations describing the two lateral bending motions in the 1storder is coinciding with the case of constant rotating speed and it was derived the explicit form of the natural frequencies, which are in very good agreement with the Campbell diagram but these frequencies are expected to be detuned in case of considering T_1 −scale amplitude modulation equations. Considering the fact that there are no periodic motions on the spinning shaft, the next step is the development of a systematic approach for the examination of the dynamics of the spinning shaft during spin-up/down operationincluding; explicit non-conservative forces, torques and imbalances of the shaft.

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