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A UNIFORM GEOMETRICAL PRESENTATION OF A QUANTUM SYSTEM EXPERIMENT

The methods of quantum physics penetrate today into various fields of theoretical disciplines. In this paper, based on quantum mechanics principles, a uniform geometric model of a quantum system experiment introduced. A phase space determined for quantum system experiment in the form of two-dimensional topological torus. On this torus, the math expectation of quantum system track defined as the wave function, along with the tensor of quantum states entanglement. The tensor of quantum entanglement interpreted as the vector basis of a local Euclidian space. The work intends the complex many-body system applications. Refs.: 22 titles.

Keywords: geometric model; quantum system; phase space; tensor of quantum states.

The problem statement. The simulation of surrounding world is, apparently, the number one attribute of intelligent creature inherent to either living matter or artificial machines. This ability appears in simple cells and reaches its height in the modern digital society. Another intrinsic feature of intelligence is communication ability via specific language of symbols. The higher intelligence level is, the richer language becomes. The third key property of intelligence is building communities of interacting cells united by common goals and rules.

The possession of the aforesaid properties transforms an arbitrary manifold of common type intelligent cells into an integrated system. A set of living cells forms a hierarchy of organs and subsystems of the body, which, in turn, acts as an element of a higher rank system (swarm, family, etc.). In general, the intellect as a phenomenon of biological or artificial nature has a multi-level hierarchical character. The law of hierarchy was discovered in ancient times and found its expression in many religious cultures.

Among the known communication languages, the Math language is, perhaps, the most concise, capacious and abstract. Like other languages, the Math arose from experience. Astrology and architecture inspired ancient sages to develop geometry. The study of mechanics brought the infinitesimal calculus. The Math development until the end of the 19-th century had been accompanied by empirical observation, while matter objects of interest and observation tools were commensurable. At that time, the differential and integral calculus had been developed, along with classical functional analysis. These disciplines were based on the concepts of absolute space and time embodied in continuum hypothesis of real numbers.

The Cantor's set theory became the common platform of mathematical models of that time, though sharp discussions took place among the scientists. On this background, the three-dimensional Cartesian space was transformed to multidimensional Euclidean space. The latter one, in turn, expanded into the complex number domain and the non-Euclidean tensor geometry on smooth manifolds. The concept of Hilbert vector space (admitting generalization to infinite case) played an important role in development the physics of small particles (quanta) in the first half of the 20th century.

Empirical investigations in small particles realm triggered new ideas and principles that contradicted known physical theories and their mathematical description. The statistical uncertainty of experimental observations, the lack of solid identification the small particles, the relativity of matter state monitoring from a subjective point of view, the ill posed "instant" particle state-to-impulse determination, made a lot perplexity in academia. The facts of quantization of the measured values did not fit the classical insight of the continual space and time. The many-body interaction caused a problem with quantum correlations (entanglement). As a consequence, the wave function of the system turned utterly cumbersome to analytical calculations. Experimental physics in the first half of the 20th century desperately needed reorganization the Math foundations to advance the theory of interactions.

The founder of the new mathematical theory of quantum mechanics became the outstanding mathematician of the 20th century, John von Neumann (1903 – 1957). Collaborating with adherents, von Neumann developed the multi-rank set theory and the special sections of functional analysis that absorbed new physical ideas and principles. These and other results became the theoretical foundation of modern quantum physics. The methods of quantum physics penetrate today into related sciences (quantum information theory and coding, biology, medicine, sociology, linguistics, etc.). However, the academic Math discipline still keeps being rather conservative and heavily relies on classical approaches.

This paper aims a cognitive insight on quantum mechanics to aid a multipolar object investigation.

Related publications survey. The genesis of quantum mechanics aids better understanding its role in the modern system analysis. The quantum mechanics was originated from the "matrix mechanics" of W. Heisenberg, M. Born and P. Jordan ([1], 1925) and "wave mechanics" Erwin Schrödinger ([2], [3], 1926). Later, von Neumann showed the equivalence between the wave mechanics built in terms of integral operators and matrix mechanics, ([4], 1932).

However, the new facts of experimental physics were still challenging the classical mathematics based on Cantor`s set theory ([5], 1883). An impressive contribution to Math foundation of quantum physics was made by

J. von Neumann. In 1927-29 Neumann published seven articles on math foundation of quantum mechanics; one of them originated new set theory ([6], 1928). These results were generalized by Neumann in his monograph ([4], 1932). Subsequent works by J. von Neumann, C. Gödel, and P. Bernays resulted in a novel axiomatic set theory, aka NBG, ([7], 1937). The Neumann's approach to set theory is distinguished by ranking the sets by ordered classes. That understanding tunes the modern view on systems hierarchy in nature and human life. The hierarchy of classes helps to eliminate known logical contradictions inherent to Cantor "naive sets theory", as well as enables construction of high order grammars with limited complexity at each level.

Authoritative experts consider Neumann's works still being the most rigorous apparatus of quantum physics, although rather cumbersome and complex ([8], 2018). Apparently, this complexity is paid for aspiration to harmonize the Hilbert function analysis with the finite scale reality [9]. The comprehensive study of this issue covers entire section of Neumann's quantum mechanics. Ultimately, a new math discipline was developed by Neumann, aka spectral theory of linear operators in Hilbert space [10].

The key point of Neumann's quantum mechanics focuses the so called eigenvalue problem [11]. This problem arises whenever inverse task of mathematical physics studied for algebraic, differential or integral equations. An accurate solution the eigenvalue problem enables major equation resolution, as well as the wave function of a quantum system acquisition [12]. A non-zero defined Hermitian operator denoted by Neumann as "maximal operator", and the eigenvalue well-posed maximal operator called "hyper maximal operator".

Hermitian operators, which are not hyper maximal, stay beyond the Neumann theory of quantum mechanics. The matter of this "exception" may be understood from Gödel's theorem on the incompleteness of any formal system [13]. Though the complete equivalence of Hilbert functional space and finite-dimensional vector space not finally proved, the operators in context of Hermitian matrix transforms became principal objects in Neumann`s quantum mechanics.

A general insight on mathematical models was presented by Neiman in his "Mathematician", ([14], 1947). In this work, geometry emphasized as a principal way of mathematical imagination and abstract thinking. Particular discussed the cognitive relationships between the "logical rigor" and "empirical knowledge" in mathematical theories.

A solid contribution in quantum mechanics made by Feynman's lectures on physics [15]. The eighth chapter of this book highlights the Hamiltonian matrix in the context of particle quantum states presentation. The Chapter 20 outlines the operator calculus as the next step in quantum mechanics understanding. The key issues in modern quantum mechanics related to measuring system and gauge fields [16 – 18]. The ubiquitous penetration of quantum physics inspired researches on general theory of the open quantum systems [19]. These works correlate with relativistic probability concept [20], [21]. As noted in [22], the quantum mechanics motivates searching for a new "quantum mathematics".

In recent years, the modern mathematics apparently migrates towards artificial and computer intelligence as a powerful driver of prospective technological solutions. In this context, the math language itself gradually evolves to high-level programming language for cognitive automata. Though, a lot of work is still ahead and more researches needed in this realm.

Objective. This paper aims constructing a unified geometric presentation of a quantum system experiment.

The quantum system hierarchy. Consider the "many-body problem" on the basis of quantum mechanics principles. Suppose a cognitive subject monitoring the object particles interaction (denote "*quantum system S experiment", or QSE*). Let experiment QSE contains a series of *cyclic tracks,* each formed by a sequence of *system state samples* (SSS). Each sample includes records of one or more *particles* of the object. An ordered set of objet particles denote "*ensemble*". Thus, we obtain the following three-level hierarchy of the QSE-terms:

− Particle ξ_n: an element of the 1-st rank set;
− Ensemble $\{\xi_n\}$ (of particles): the 1-st rank set;
− Sample (of particles) $x \subseteq \{\xi_n\}$: the 1-st rank instant subset;
− Track $f(t) = \{x_t\}$: the 2-nd rank sequence of samples;
− Experiment: the 3-rd rank collection of tracks;
– Universe $U = \{u(t)\}$: the 3-rd rank collection of potential tracks.

We adopt the following axioms of quantum system experiment (QSE).
1) Separability. Any particle appears solo once at least.
2) Duality. Any particle has an antiparticle dual to itself.
3) Ergodicity. Each track of QSE observed under the same conditions.
4) Cyclicity. Each track starts and ends with the same state.

Let $\hat{f}(t) \in U$ − the mathematical expectation of a track; $\{\tilde{f}(t)\} \in U$ − the set of centered tracks. Consider $\tilde{f}(t)$ a Markov chain; $P(k,l)$ − transition matrix for QSE (entanglement).

Consider symmetric real Hermitian matrix $P(k,l)$. Each row of $P(k,l)$ presents the complete set of conditional probabilities for QSE-states

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transition. Therefore, the sum of all the elements in any row of $P(k,l)$ matrix yields unit.

Introduce axiom 5. The QSE states Separability.

Each diagonal element of $P(k,l)$ *i* s greater than zero. It means that any QSE-state appears solo once at least. Thus, any non-diagonal element of $P(k,l)$ is less than unit. Map $P(k,l)$ onto Hermitian matrix $H(k,l)$ by replacing the diagonal of $P(k,l)$ with units; apparently this transformation is a bejective mapping: $P(k,l) \leftrightarrow H(k,l)$.

Presentation Hermitian matrix in eigenvalue spectrum. The quantum system experiment (QSE) itself is an empirical data array beyond any formalized coordinate system of a physical realm. Constructing an abstract math space, which is relevant to given QSE, is the essence of a particular quantum model.

Take a well-posed Hermitian matrix $H(k,l)$, which admits an accurate eigenvalue task solution with real non-zero spectrum of eigenvalues $\{\lambda_k\}$ and unitary real matrix Z of eigenvectors. Let $\Lambda = diag(\{\lambda_k\})$ the diagonal matrix of eigenvalues. For these notations, the eigenvalue task is known equation:

$$H \cdot Z^* = Z^* \cdot \Lambda, \tag{1}$$

where Z^* is the conjugate matrix towards Z. Let I the diagonal unitary matrix. The following relation is valid for any unitary matrix Z:

$$Z \cdot Z^* = Z^* \cdot Z = I. \tag{2}$$

Evolve the (2) equation as follows:

$$Z : (H \cdot Z^*) \cdot Z = (Z^* \cdot \Lambda) \leftrightarrow H \cdot (Z^* \cdot Z) = (Z^* \cdot \Lambda) \leftrightarrow H \cdot I = (Z^* \cdot \Lambda).$$

This yields the known equation presenting a well-posed Hermitian matrix H in the basis of eigenvectors Z:

$$H = Z^* \cdot \Lambda \cdot Z. \tag{3}$$

The presentation (3) possesses the property

$$tr(H) = tr(\Lambda) = \sum_{k=1}^{K} \lambda_k = K, \tag{4}$$

where $tr(\ )$ is matrix treasure (the sum of diagonal elements).

Geometric presentation of the quantum system experiment. The core idea of geometric presentation the matrix of quantum states transitions $P(k,l) \leftrightarrow H(k,l)$ is the fact that any well-posed Hermitian matrix H corresponds to particular set of vectors $\vec{V} = \{V_k\}$ in Euclidian space.

Apply the aforesaid number 4 axiom of quantum experiment Cyclicity to introduce the notion of topological time circle Θ_T, and the so called "topological time" $t \in \langle \Theta_T = [0, 1, 2, ..., T = 0] \rangle$ as a closed chain of time dots. To each point t assign an ordered sample $x_t \subseteq \{\xi_n\}$ of the quantum system state, which is an integer signed number s formed by N ternary digits (any digit refers to particular quantum particle $\xi_n \in \{\xi_n\}$). In case $N = 8$, numbers $\xi_{\min} = -11111111$ and $\xi_{\max} = +11111111$ present the minimal and maximal elements of the set $\xi_n \in \{\xi_n\}$. Next, one axiom more needed to accomplish the topological space of QSE.

Axiom 6. Completeness of the QSE set of states.

A singular state Ω of quantum system exists between the minimal and maximal states: $-111...1 < \Omega < +111...1$.

Include the Ω number into the set $\{x\}$, $x \subseteq \{\xi_n\}$: $\{x\} \to \langle \{s\}, \Omega \rangle = X$. Present the set X as topological circle Θ_X of QSE states. Thus, the state space of a quantum system takes the form of a two-dimensional torus $\Theta^2 = \Theta_X \times \Theta_T$. Each track $f(t) = \{x_t\}$ refers to a loop set on Θ^2. The math expected track $\hat{f}(t) = mean(f(t))$ is a loop on Θ^2. The torus $\Theta^2 = \Theta_X \times \Theta_T$ denote "the *phase space of quantum system states*" (PSS). The track $\hat{f}(t) \in \Theta^2$ denote "the *wave function* of QSE". Let each point of $\hat{f}(t) \in \Theta^2$ possess a matrix $H_t(k,l)$ originated from $P_t(k,l)$. The form $\langle \hat{f}(t) \in \Theta^2, H_t(k,l) \rangle$ denote "the QSE functional space (QFS)", where $H_t(k,l)$ is Riemann metric tensor of second rank.

If Markov quantum system, the tensor H_t does not depend on the time t. To calibrate the QFS space, construct the first-rank matrix operator V based on matrix H_t. Define the matrix of the so called amplitude eigenvalues spectrum $Y = \pm\sqrt{\Lambda}$. Evolve the equation (3) to

$$H = Z^* \cdot (Y \cdot Y) \cdot Z = (Z^* \cdot Y) \cdot (Y \cdot Z). \qquad (5)$$

Insert the neutral math form $Z \cdot Z^* = I$ into (5):

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$$(Z^* \cdot Y) \cdot (Y \cdot Z) = (Z^* \cdot Y) \cdot I \cdot (Y \cdot Z) =$$
$$= (Z^* \cdot Y) \cdot (Z \cdot Z^*) \cdot (Y \cdot Z) =$$
$$= (Z^* \cdot Y \cdot Z) \cdot (Z^* \cdot Y \cdot Z).$$

Denote $V := (Z^* \cdot Y \cdot Z)$. Apparently,

$$V^* = (Z^* \cdot Y \cdot Z)^* = (Z)^* \cdot Y \cdot (Z^*)^* = Z^* \cdot Y \cdot Z = V,$$

i.e. matrix V turns Hermitian one. Hence, $H = V^2$, or

$$V = \sqrt{H} = Z^* \cdot Y \cdot Z. \quad (6)$$

The Hermitian matrix V is the first-rank tensor with respect to H; it is covariant on vector set $\vec{V} = \{\vec{V}_k\}$. Assume the matrix V is a two-dimensional array of normal projections for vectors $\vec{V} = \{\vec{V}_k\}$, being presented in the orthonormal basis of an Euclidean space (considering H is the scalar product of $\vec{V} : \{\vec{V} \times \vec{V}\} = H$). Thus, matrix H admits geometric interpretation as a set of vectors (6) acquired from the experiment. However, the set of vectors V in (6) is not unique, but a plenty of different vector sets are invariant to any given Hermitian matrix H of vector scalar products. Therefore, equation (6) exhibits a *unified quantum uncertainty* principle.

Ultimately, a unified experiment on quantum system (QSE) is mapped on two-dimensional topological torus $\Theta^2 = \Theta_X \times \Theta_T$, which is built in terms of "*quantum state multiplied by quantum time*", and intends to exhibit the complete set of potential QSE-states. This torus $\Theta^2 = \Theta_X \times \Theta_T$ is understood as QSE phase state space (denoted above PSS). Next, a phase-loop track $\hat{f}(t) = mean(f(t))$ defined on the torus $\Theta^2 = \Theta_X \times \Theta_T$, to exhibit the math expected QSE-track (called QSE wave function). Finally, tensor function $V(\hat{f}_t)$ determined on the points of math expected track $\hat{f}(t)$, that exhibits the QSE particle entanglement in the form of vector set $\{\vec{V} \times \vec{V}\}_t = H_t$.

The Hermitian matrix $V(\hat{f}_t)$ is understood as the kernel of a first-rank tensor operator for quantum system states transformation (the so called operator equation of physical process observation): $x \cdot V = x_V$. Its geometric form is the vector basis of a local Euclidian space determined on the set of track points $\hat{f}(t)$. The evaluation of unknown state x turns to the known inverse task of Math physics: $x_V = x \cdot V^{-1}$, where the kernel V often ill-posed matrix assumed for renormalization procedure.

Conclusion. The theoretical methods of large system investigation evolve behind empirical researches. Beginning the 20th century a cascade of bizarre empirical facts emerged in experimental physics, resulted in modern quantum mechanics foundation. In recent years, the quantum physics penetrates ubiquitously, though, academia mathematical disciplines often keep inertia.

In this paper, the quantum principles applied to particle interaction analysis in an arbitrary many-body task. A unified geometric model designed for a quantum system experiment, based on the axiomatic approach. The topological space defined for quantum system phase states in the form of two-dimensional torus. The looped tracks on the torus simulate the system state behavior in cyclic time, as well as the wave function of quantum system determined and tensor form of particle entanglement obtained.

The results of this work oriented for a wide class of large object studies, wherein a set of interacting elements observed (telecommunication and information networks, transporting and queuing systems, etc.)

References:

1. Born, M., Heisenberg, W., and Jordan, P. (1925), "Zur Quantenmechanik II", *Zeitschrift für Physik*, No. 35, S. 557-615.

2. Schrödinger, E. (1926), *Quantisierung als Eigenwertproblem [Electronic resource].* Available: https://doi.org/10.1002/andp.19263840404.

3. Schrödinger, E. (1927), *Abhandlungen zur Wellenmechanik*, Barth, 169 p.

4. J. von Neumann, J. (1996), *Mathematische Grundlagen der Quantenmechanik*, Springer Customer Service Center GmbH, Haberstrasse 7, Heidelberg, Germany, 262 p.

5. Cantor G. (1976), "Foundations of a general theory of manifolds" (1st english translation) [Electronic resource], *Journal of the National Caucus of Labor Committees*, Available: http://wlym.com/archive/campaigner/7602.pdf.

6. J. von Neumann, J. (1928), "Die Axiomatisierung der Mengenlehre", *Mathematische Zeitschrif*, No. 27, S. 669-752.

7. Bernays, P. (1937), "A System of Axiomatic Set Theory—Part *I* ", *The Journal of Symbolic Logic,* No. 2, pp. 65-77.

8. J. von Neumann, J. (2018), *Mathematical Foundations of Quantum Mechanics*, New Edition by John von Neumann Paperback, 328 p.

9. Mejlbro, L. (2009), *Hilbert Spaces and Operators on Hilbert Spaces* [Electronic resource], Available: http://zums.ac.ir/files/research/site/ebooks/petroleum-gas-oil/hilbert-spaces-and-operators-on-hilbert-spaces.pdf.

10. Warzel, S., and Reppekus J. (2013), *Spectral Theory in Hilbert Spaces* [Electronic resource] Available: https://www-m7.ma.tum.de/foswiki/pub/M7/Analysis/VorSpecTheory/SpectralTh.pdf.

11. Sinayoko, S. (2012), *Eigenvalue problems I: Introduction and Jacobi Method* [Electronic resource] Available: http://www.southampton.ac.uk/~feeg6002/lecturenotes/feeg6002_numerical_ methods08.pdf.

12. Lasanen, S. (2014), *Introduction to inverse problems* [Electronic resource] Available: http://cc.oulu.fi/~smaenpaa/inv_pk/inv_pk14_eng.pdf.

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13. Cellucci, C. (1992), "Gödel's incompleteness theorem and the philosophy of open systems", *Travaux de logique*, No.7, P. 103-127.
14. Domokos, S. (2011), John von Neumann, the Mathematician [Electronic resource], Available: http://math.bme.hu/~szasz/files/neumann.pdf.
15. Feynman, R., Leighton, R., and. Sands M. (2013), The Feynman Lectures on Physics, Volume III [Electronic resource], Available: http://www.feynmanlectures.caltech.edu/III_toc.html.
16. Vatsya, S.R. (2017), "Current State of Quantum Theory", *Physics & Astronomy International Journal,* Vol. 1, Issue 4, pp. 1-8.
17. Carloni, L. (2011), Renormalization in Effective Field Theory and Hidden Radiation [Electronic resource], Available: http://portal.research.lu.se/ws/files/5611928/1833297.pdf.
18. Gripaio,s B. (2016), Gauge Field Theory [Electronic resource], Available: http://www.hep.phy.cam.ac.uk/~gripaios/gft_lecture_notes.pdf.
19. Breuer, H., Petruccione, F. (2007), The Theory of Open Quantum Systems [Electronic resource], Oxford Scholarship Online, Available: http://www.oxfordscholarship.com/view/10.1093/acprof:oso/9780199213900.001.0001/acprof-9780199213900.
20. Cresser, J. (2009), Observables and Measurements in Quantum Mechanics [Electronic resource], Available: http://physics.mq.edu.au/~jcresser/Phys301/Chapters/Chapter13.pdf.
21. Khrennikov, A. (2016), *Probability and Randomness. Quantum versus Classical*. Imperial College Press, London, 282 p.
22. Dijkgraaf, R. (2017), Quantum Questions Inspire New Math [Electronic resource], Quanta Magazine, Available: https://d2r55xnwy6nx47.cloudfront.net/uploads/2017/03/how-quantum-theory-is-inspiring-new-math-20170330.pdf.

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Методи квантової фізики сьогодні проникають у різні галузі теоретичних дисциплін. В роботі, заснованої на принципах квантової механіки, введено рівномірну геометричну модель експерименту квантової системи. Фазовий простір визначається для експерименту квантової системи у вигляді двовимірного топологічного тору. На цьому торі, математичне очікування квантової системи слід визначається як хвильова функція поряд з тензором квантових станів завалення. Тензор квантового заплутування трактується як векторна основа локального евклідового простору. Робота передбачає комплексне застосування багатьох систем. Бібліогр.: 22 назв.

Ключові слова: геометрична модель; квантова система; фазовий простір; тензор стану.

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Обобщенное геометрическое представление квантового системного эксперимента / **Тихонов В.И**. // Вестник НТУ "ХПИ". Серия: Информатика и моделирование. – 2018. – №. 24 (1300). – Р. 59 – 68.

Методы квантовой физики сегодня проникают в различные области теоретических дисциплин. В этой статье, основанной на принципах квантовой механики, введена однородная геометрическая модель эксперимента квантовой системы. Фазовое пространство, определенное для квантового системного эксперимента в виде двумерного топологического тора. На этом торе математическое ожидание квантовой траектории системы, определяемое как волновая функция, наряду с тензором сцепления квантовых состояний. Тензор квантовой запутанности интерпретируется как векторный базис локального евклидова пространства. Эта работа предполагает комплексные приложения для многих систем. Библиогр.: 22 назв.

Ключевые слова: геометрическая модель; квантовая система; фазовое пространство; тензор состояния.

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A uniform geometrical presentation of a quantum system experiment / **Tikhonov V.I.** // Herald of the National Technical University "KhPI". Subject issue: Information Science and Modeling. – 2018. – №. 24 (1300). – P. 59 – 68.

The methods of quantum physics penetrate today into various fields of theoretical disciplines. In this paper, based on quantum mechanics principles, a uniform geometric model of a quantum system experiment introduced. A phase space determined for quantum system experiment in the form of two-dimensional topological torus. On this torus, the math expectation of quantum system track defined as the wave function, along with the tensor of quantum states entanglement. The tensor of quantum entanglement interpreted as the vector basis of a local Euclidian space. The work intends the complex many-body system applications. Refs.: 22 titles.

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