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Application of R-Functions Theory to Study Parametric Vibrations and Dynamical Stability of Laminated Plates

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Abstract
The problem of nonlinear parametric vibrations and stability analysis of the symmetric laminated plates is considered. The proposed method is based on multimode approximation of displacements and solving series auxiliary linear tasks. The main feature of the work is the application of the R-functions theory, which allows investigating parametric vibrations of plates with complex shape and different boundary conditions.

Keywords
R function method, laminated plates, parametric vibrations

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Introduction

This work is devoted to a study of the nonlinear vibrations and stability of laminated plates with complex geometric shape that are subjected to a periodic in-plane load. The relevance of the problem is explained by wide adoption of composite materials in the industrial applications. A special attention has been paid to the vibrations of composite plates under various types of loading, and in particular, parametric vibrations. There are many publications on this subject, but the previous works considered mostly the plates of a canonical form with a homogeneous subcritical state. Currently, the computer simulation of the nonlinear dynamics of plates with complex geometric shape and inhomogeneous subcritical state are performed using the Finite Element Method (FEM) [5]. A different approach has been proposed in references [2,4]. It is based on the theory of R-functions and variational methods, and enables obtaining the meshless solutions to the plate and shell vibration problems. In this work the R-functions method (RFM) is extended to a new class of problems – nonlinear parametric vibrations and dynamical stability of laminated plates. In the proposed approach we will take into account the subcritical state.

1. Mathematical Statement

Let us consider the parametric vibrations of the laminated plates with symmetric structure. We assume that plate and all its layers have a constant thickness; and the plate is subjected to a periodic in-plane load $p = p_0 + p_t \cos\theta t$, where p_0 is a static component, p_t is an amplitude of a periodic part, and θ is a frequency of the load. We derive mathematical formulation of the problem using Kirchhoff’s hypotheses. Accordingly, the strains acting in the midplane are expressed as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\ \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},$$

$$\chi_x = -\frac{\partial^2 w}{\partial x^2};\ \chi_{xy} = -2\frac{\partial^2 w}{\partial x \partial y};\ \chi_y = -\frac{\partial^2 w}{\partial y^2}.$$

In these expressions *u, v, w* are the displacements of the points in a midplane in the directions of the coordinate axes *Ox, Oy*, and *Oz*. Stress resultants N_x, N_y, N_{xy} and moments M_x, M_y, M_{xy} are presented as:

$$\{N\} = [\mathbf{C}] \cdot \{\varepsilon\},\quad \{M\} = [\mathbf{D}] \cdot \{\chi\},$$

where **C** and **D** are stiffness matrices:

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{pmatrix},$$

$\{\varepsilon\}, \{\chi\}, \{N\}, \{M\}$ are strain, stress and moment vectors:

$$\{\varepsilon\} = (\varepsilon_x, \varepsilon_y, \varepsilon_{xy})^T,\quad \{\chi\} = (\chi_x, \chi_y, \chi_{xy})^T,$$
$$\{N\} = (N_x, N_y, N_{xy})^T,\quad \{M\} = (M_x, M_y, M_{xy})^T.$$

The components C_{ij}, D_{ij} $(ij = 11,22,12,16,26,66)$ of stiffness matrices are defined as [1]:

$$(C_{ij}, D_{ij}) = \sum_{s=1}^{N} \int_{h_s}^{h_{s+1}} B_{ij}^{(s)}(1, z^2)dz,\quad i,j = 1,2,6,$$

where $B_{ij}^{(s)}$ are the mechanical characteristics of the s-*th* layer. Ignoring the inertial terms, the motion equations, expressed in the displacements, can be written as follows [1]:

$$L_{11}u + L_{12}v = -Nl_1(w),\ L_{21}u + L_{22}v = -Nl_2(w) \tag{1}$$

$$L_{33}w = Nl_3(u,v,w) - \varepsilon m_1 \frac{\partial w}{\partial t} - m_1 \frac{\partial^2 w}{\partial t^2} \tag{2}$$

Differential operators L_{ij}, Nl_i $i,j = 1,2,3$ in the equations (1)-(2) are defined as

$$L_{11}w = C_{11}\frac{\partial^2}{\partial x^2} + 2C_{16}\frac{\partial^2}{\partial x \partial y} + C_{66}\frac{\partial^2}{\partial y^2},\ L_{22} = C_{66}\frac{\partial^2}{\partial x^2} + 2C_{26}\frac{\partial^2}{\partial x \partial y} + C_{22}\frac{\partial^2}{\partial y^2},$$

$$L_{12} = L_{21} = C_{16}\frac{\partial^2}{\partial x^2} + (C_{12} + C_{66})\frac{\partial^2}{\partial x \partial y} + C_{26}\frac{\partial^2}{\partial y^2},$$

$$L_{33} = D_{11}\frac{\partial^4}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4}{\partial x^2 \partial y^2} + 4D_{16}\frac{\partial^4}{\partial x^3 \partial y} + 4D_{26}\frac{\partial^4}{\partial x \partial y^3} + D_{22}\frac{\partial^4}{\partial y^4},$$

$$Nl_1 = \frac{\partial w}{\partial x}L_{11}w + \frac{\partial w}{\partial y}L_{12}w,\ Nl_2 = \frac{\partial w}{\partial x}L_{12}w + \frac{\partial w}{\partial y}L_{22}w,$$

$$Nl_3 = N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y},$$

where ε is a damping coefficient. System (1)-(2) is supplemented by the initial and boundary conditions. The load, specified at the traction portion of the plate's boundary, is specified by its normal and tangential components:

$$N_n = -p, T_n = 0 .$$

The normal component N_n of the applied load can be expressed via the stress resultants N_x, N_y, N_{xy}

$$N_n = N_x l^2 + N_y m^2 + 2N_{xy} lm,\ T_n = N_{xy}\left(l^2 - m^2\right) + \left(N_y - N_x\right) lm ,$$

where $l = \cos(n, Ox), m = \cos(n, Oy)$ are directional cosines of the normal vector n to the plate's boundary.

2. Investigation Method

The proposed method reduces solution of a nonlinear problem to a series of auxiliary linear problems. First, we need to determine the subcritical state and solve a linear vibration problem for the loaded plate in the midplane. Detailed description of the solution methods involved in this step as well as several numerical examples can be found in the reference [4]. Once eigenfunctions w_i of a linear vibration problem are determined, they can be utilized in a truncated series to represent the deflection w of the plate:

$$w(x, y, t) = \sum_{i=1}^{n} y_i(t) w_i(x, y). \tag{3}$$

To satisfy the motion equations (1) we propose to present the in-plane displacements as

$$u(x, y, t) = u_1(x, y)p + \sum_{i,j=1}^{n} y_i(t) y_j(t) u_{ij}(x, y),\quad v(x, y, t) = v_1(x, y)p + \sum_{i,j=1}^{n} y_i(t) y_j(t) v_{ij}(x, y). \tag{4}$$

In these expressions functions (u_1, v_1) are solution of system

$$L_{11}u_1 + L_{12}v_1 = 0,$$
$$L_{21}u_1 + L_{22}v_1 = 0,$$

supplemented by the following boundary conditions:

$$N_n^{(L)}(u_1, v_1) = -1,\ T_n^{(L)}(u_1, v_1) = 0,$$

on the loaded part of the domain's boundary $\partial\Omega_1$. Here $N_n^{(L)}$, $T_n^{(L)}$ are linear parts of N_n, T_n :

$$N_n = N_n^{(L)} + N_n^{(N)}, \qquad T_n = T_n^{(L)} + T_n^{(N)},$$
$$N_n^{(L)} = \{P\}\cdot\{\varepsilon\}^{(L)},\ N_n^{(N)} = \{P\}\cdot\{\varepsilon\}^{(N)}\ T_n^{(L)} = \{Q\}\{\varepsilon\}^{(L)},\ T_n^{(N)} = \{Q\}\{\varepsilon\}^{(N)}.$$

Vectors $\{P\} = (P_1, P_2, P_3)$ and $\{Q\} = (Q_1, Q_2, Q_3)$ are defined as:

$$P_1 = C_{11}l^2 + C_{12}m^2 + 2C_{16}lm;\ P_2 = C_{12}l^2 + C_{22}m^2 + 2C_{26}lm;$$
$$P_3 = C_{16}l^2 + C_{26}m^2 + 2C_{66}lm;$$
$$Q_1 = C_{11}(l^2 - m^2) + (C_{12} - C_{11})lm;\ Q_2 = C_{26}(l^2 - m^2) + (C_{22} - C_{12})lm;$$
$$Q_3 = C_{66}(l^2 - m^2) + (C_{26} - C_{16})lm.$$

Functions (u_{ij}, v_{ij}) are solutions of the following inhomogeneous system:

$$L_{11}(u_{ij})+L_{12}(v_{ij})=-Nl_1^{(2)}(w_i\,,w_j),$$
$$L_{21}(u_{ij})+L_{22}(v_{ij})=-Nl_2^{(2)}(w_i\,,w_j), \tag{5}$$

They satisfy the following boundary conditions:

$$N_n^{(L)}(u_{ij},v_{ij})=-N_n^{(N)}(w_i,w_j),\;\; T_n^{(L)}(u_{ij},v_{ij})=-T_n^{(N)}(w_i,w_j). \tag{6}$$

Right hand side of the system (5) depends on the eigenfunctions w_i and can be written as follows:

$$Nl_1^{(2)}\left(w_i^{(c)},\,w_j^{(c)}\right)=w_{i,x}L_{11}w_j+w_{i,y}L_{12}w_j\,,$$
$$Nl_2^{(2)}\left(w_i^{(c)},\,w_j^{(c)}\right)=w_{i,x}L_{12}w_j+w_{i,y}L_{22}w_j\,.$$

Functions in the right hand side of the boundary conditions (6) are defined as

$$N_n^{(N)}(w_i,w_j)=\frac{1}{2}\left(P_1\frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial x}+P_2\frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial y}+\;P_3\left(\frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial x}+\frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial y}\right)\right),$$
$$T_n^{(N)}(w_i,w_j)=\frac{1}{2}\left(Q_1\frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial x}+Q_2\frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial y}+\;Q_3\left(\frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial x}+\frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial y}\right)\right).$$

To determine the functions u_1, v_1, u_{ij}, v_{ij} we will use the R-function method (RFM).

Substituting expressions (3),(4) in equation (2) and applying Bubnov-Galerkin method to the resulting equation, we will obtain a system of ordinary differential equations (ODEs):

$$y_m''(t)+\varepsilon y_m'(t)+\Omega_m^2\left(y_m(t)+p_t\cos\theta t\sum_{k=1}^{n}\alpha_k^{(m)}y_k(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{ijk}^{(m)}y_i(t)y_j(t)y_k(t)\right)=0\,, \tag{7}$$

where $m=\left(\overline{1,n}\right)$. The coefficients in this system of ODEs are defined by the formulas:

$$\alpha_k^{(m)}=-\frac{\iint\limits_{\Omega}Nl_{31}(u_1,v_1,w_k)w_m d\Omega}{m_1\Omega_m^2\left\|w_m\right\|^2},\;\;\gamma_{ijk}^{(m)}=-\frac{\iint\limits_{\Omega}Nl_{32}(u_{ij},v_{ij},w_k)w_m d\Omega}{m_1\Omega_m^2\left\|w_m\right\|^2},$$

where

$$Nl_{31}(u_1,v_1,w_k)=N_x^{(L)}\frac{\partial^2 w_k}{\partial x^2}+\;N_y^{(L)}\frac{\partial^2 w_k}{\partial y^2}+2N_{xy}^{(L)}\frac{\partial^2 w_k}{\partial x\partial y},$$
$$Nl_{32}(u_{ij},v_{ij},w_k)=N_x\frac{\partial^2 w_k}{\partial x^2}+\;N_y\frac{\partial^2 w_k}{\partial y^2}+\;2N_{xy}\frac{\partial^2 w_k}{\partial x\partial y}.$$

Let us consider in detail the one-mode approximation:

$$w(x,y,t)=y(t)\,w_1(x,y)$$
$$u(x,y,t)=p(t)u_1(x,y)+y^2(t)\cdot u_{11}(x,y),\;\; v(x,y,t)=p(t)v_1(x,y)+y^2(t)\cdot v_{11}(x,y),$$

In this case the system of equations (7) is reduced to one equation

$$y''(t)+\varepsilon y'(t)+\Omega_L^2(1+\alpha p_t\cos\theta t+\gamma\,y^2(t))y(t)=0\,. \tag{8}$$

Equation (8) uses the following notations:

$$y(t) = y_1(t),\ \Omega_L = \Omega_1,\ \alpha = \alpha_1^{(1)},\ \gamma = \gamma_{111}^{(1)}.$$

Equation (8) can be transformed to the known form [3]:

$$y''(t) + 2\varepsilon_1 y'(t) + \Omega_L^2((1 - 2k\cos\theta t)y(t) + \gamma y^3(t)) = 0, \tag{9}$$

where $\alpha \cdot p_t = -2k,\ \varepsilon_1 = \varepsilon/2$.

The main task of investigation of parametric vibrations is finding instability areas and studying behavior of plate after loss of stability. To investigate the stability [3], it is enough to consider the linearized equation ($\gamma = 0$): it is well-studied Mathieu equation and its main instability domain is situated near $\theta = 2\Omega_L$ and bounded by curves [3]:

$$2\Omega_L\sqrt{1 - \sqrt{k^2 - \left(\frac{\Delta}{\pi}\right)^2}} \le \theta \le 2\Omega_L\sqrt{1 + \sqrt{k^2 - \left(\frac{\Delta}{\pi}\right)^2}},$$

where $\Delta = \dfrac{2\pi\varepsilon_1}{\Omega}$. The relation between the frequency ratio and the amplitude of nonlinear vibrations after the loss of stability, according to [3], has the form:

$$A = \frac{2}{\sqrt{3\gamma}}\sqrt{\left(\frac{\theta}{2\Omega_L}\right)^2 - 1 \pm \sqrt{k^2 - \left(\frac{\theta}{2\Omega_L}\right)^2\left(\frac{\Delta}{\pi}\right)^2}}.$$

3. Parametric vibrations of a plate with circular cutouts

Now we will use the proposed method to investigate parametric vibrations of a three-layered plate with circular cutouts that is shown in Fig.1. Plate is subjected to a load along the sides parallel to axis *OX*.

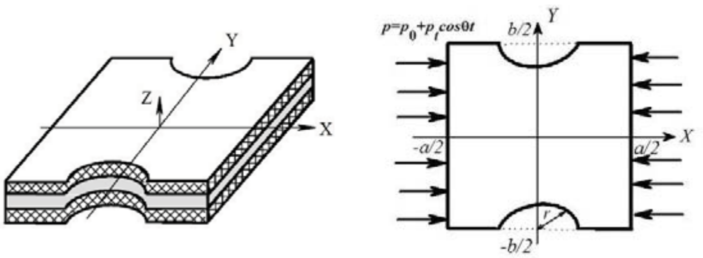

Figure 1

Numerical results are obtained for the following mechanical parameters (glass-epoxy $E_1/E_2 = 3,\ G/E_2 = 0.6,\ \nu_1 = \nu_2 E_1/E_2 = 0.25$) and geometric parameters ($b/a = 1,\ 2r/a = 0.5,\ h/a = 0.01$). The boundary of the plate is simply supported.

Table 1 presents the values of the frequency and critical load parameters

$$\bar{\lambda} = \Omega_L a^2\sqrt{\frac{12(1-\mu_1\mu_2)}{E_2 h^2}},\ \bar{p}_{kr} = a^2 p_{kr}/E_2 h^3.$$

Table 1. Frequency and critical load parameters

$\bar{p}_{kr}$	$\bar{\lambda}$		
	p_0/p_{kr}		
	0.25	0.5	0.75
9.45	46.68	38.33	27.29

Results of the influence analysis of static load component on the location of the instability domains and amplitude-frequency relations are presented in Fig. 2,3. The load increase results in the shift of domains to lower values of excitation frequency. The value of the static component of load affects the slope of the curves.

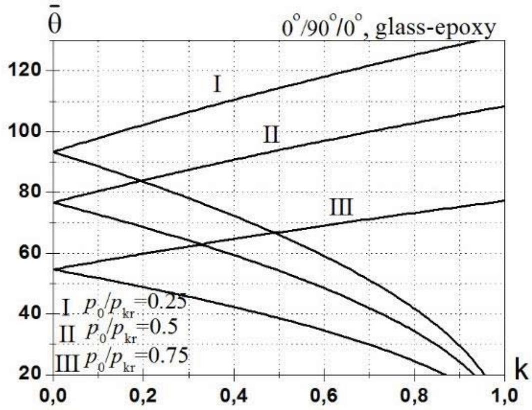

Figure 2

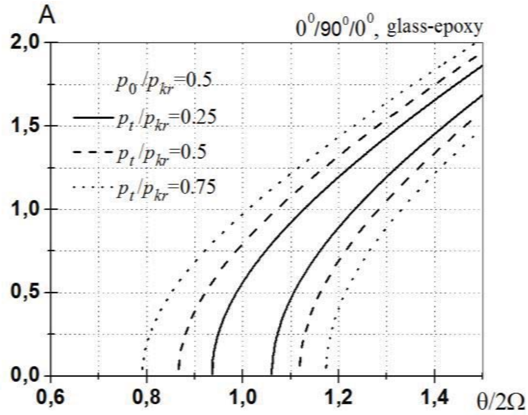

Figure 3

Conclusions

The paper presents an approach for studying the parametric vibrations of the laminated plates with symmetric structure and constant thickness. Applying multi-mode approximation enables investigating the parametric resonances of the plate near $\theta = 2\Omega_i, i = 1,2,..$ and mutual influence of the vibration modes. The method is based on the theory of R-functions, which makes this method useful for plates of complex geometric shape and various boundary conditions.

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